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Comments on 1/16 BPS quantum states and classical configurations

Lars Grant,[ab] **Pietro A. Grassi,**[c] **Seok Kim**[de] **and Shiraz Minwalla**[a]

[a] *Department of Theoretical Physics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai - 400005, India*
[b] *Jefferson Lab, Department of Physics, Harvard University, Cambridge, MA 02138, U.S.A.*
[c] *Department of Sciences and Advanced Technologies & INFN, University of Eastern Piedmont, Alessandria, I-15100, Italy*
[d] *Theoretical Physics Group, Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.*
[e] *Institute for Mathematical Sciences, Imperial College, London SW7 2PE, U.K.*
E-mail: lars.grant@mail.mcgill.ca, pgrassi@cern.ch, s.kim@imperial.ac.uk, minwalla@theory.tifr.res.in

Abstract: We formulate the problem of counting 1/16 BPS states of $\mathcal{N}=4$ Yang Mills theory as the enumeration of the local cohomology of an operator acting on holomorphic fields on C^2. We study aspects of the enumeration of this cohomology at finite N, especially for operators constructed only out of products of covariant derivatives of scalar fields, and compare our results to the states obtained from the quantization of giant gravitons and dual giants. We physically interpret the holomorphic fields that enter our conditions for supersymmetry semi-classically by deriving a set of Bogomolnyi equations for 1/16-BPS bosonic field configurations in $\mathcal{N}=4$ Yang Mills theory on R^4 with reality properties and boundary conditions appropriate to radial quantization. An arbitrary solution to these equations in the free theory is parameterized by holomorphic data on C^2 and lifts to a nearby solution of the interacting Bogomolnyi equations only when the constraints equivalent to Q cohomology are obeyed.

Keywords: AdS-CFT Correspondence, Black Holes in String Theory, Supersymmetric gauge theory.

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Contents

1. Introduction

As $\mathcal{N}=4$ Yang Mills theory is dual to IIB string theory on $AdS_5\times S^5$ [1−4], exact results in this theory hold the promise of important lessons about gravitational dynamics. A quantity that appears physically interesting as well as potentially calculable is the exact finite N partition function over supersymmetric states in this Yang Mills theory quantized on S^3. In the bulk such a partition function would characterize an interacting collection of supersymmetric gravitons, giant gravitons [5−7] and black holes at successively higher energies, and should contain a wealth of information about the structure and dynamics of these objects. This BPS partition function may also turn out to have mathematical interest.

The partition function over states that preserve at least one eighth of the supersymmetries of Yang Mills theory has already been determined [8]. Related issues are discussed in [9, 10]. Moreover the most general index over all supersymmetric states in this theory has also been determined in terms of an integral over a single unitary matrix (at finite N) [8]. However the full finite N partition function over all supersymmetric states - which includes contributions from 1/16 BPS states - has yet to be found. As IIB supergravity on $AdS_5\times S^5$ hosts no regular 1/8 BPS black holes but (at least) a four parameter class of 1/16 supersymmetric black hole solutions [11] (see [12−15] also), the full supersymmetric partition function is likely to be qualitatively richer than those that have been determined to date.

In this paper we report modest progress towards characterizing the 1/16 BPS partition function of $\mathcal{N}=4$ Yang Mills theory, which we now proceed to describe.

Our paper is based on the following conjecture. We conjecture that the supersymmetric spectrum of $\mathcal{N}=4$ Yang Mills theory on S^3 is exactly given by the spectrum of 1/16 BPS states of the finite N one loop Beisert Hamiltonian of $\mathcal{N}=4$ Yang Mills [16, 17]. This conjecture (which is implicit in [8] and [18] and was explicitly stated in [19]) is as yet unproved but is consistent with all known data about the supersymmetric spectrum of Yang Mills theory, including strong-coupling countings [8, 20, 21]. Via the state operator map, it is equivalent to the statement that the 1/16 BPS spectrum of $\mathcal{N}=4$ Yang Mills is given by the nonlinear classical cohomology of a particular supercharge in the supersymmetry algebra (see section 2.2).

In section 2 of this paper we describe the classical action of this special supercharge on 'letters' of Yang Mills theory at the origin of R^4. These 'letters' are simply all covariant derivatives of the basic gauge invariant fields - scalars, field strengths and fermions - of $\mathcal{N}=4$ Yang Mills theory (subject to the operator equations of motion). The structure of the action of a supercharge, Q, on any letter A of Yang Mills takes the form $QA=[B,C]$

where B and C are other letters, and the commutator involves matrix multiplication. It follows immediately from this structure that Q acts on products of traces 'one at a time'. This allows us to argue that the cohomology of Q at energies smaller than N (when products of traces form an unambiguous basis of gauge invariant operators) is simply given by the Bose Fermi multiparticling of the single trace Q cohomology.

This result, which follows immediately from general considerations, already makes an interesting prediction for bulk dynamics. Recall that single trace operators are dual to bulk gravitons. Consider a collection of bulk gravitons, each of which is individually annhihilated by a particular supercharge. Our result implies that the collection in question is also annihilated by the same supercharge (i.e. that $1/N$ interaction effects do not lift the energy of this collection above the sum of energies of its consitutents) provided that the sum of energies of the gravitons is smaller than N. We do not know how to derive this result from the bulk, but it follows immediately as a consequence of our conjecture about Q cohomology. As the set of 1/16-BPS single trace Yang Mills operators is already known (it consists of descendants of chiral primaries), our argument lends support to the 1/16 BPS partition function of Yang Mills theories at energies smaller than N conjectured in [8] (see also [22]).

In section 2 we also demonstrate that Q simplifies when acting on a local field which may mathematically be thought of as a generating function for covariant derivatives of the letters of Yang Mills theory. We will find a physical interpretation of this generating function in section 5.

Having listed the action of the Q operator, it is natural to attempt to ennumerate its cohomology. We have achieved very modest progress in this direction. Our concrete results pertain to special subsectors of Yang Mills theory. First consider a subsector of operators built out of any number of supersymmetric covariant derivatives of a single supersymmetric scalar. It turns out to be easy to evaluate the partition function over all operators in Q cohomology which have at least one representative in this sector. Our result in fact agrees with the partition function previously obtained by Mandal and Suryanarayana by quantizing a related class of giant gravitons [21]. We point out that the naive quantization of the relevant dual giant gravitons does not reproduce the same result, and provide a physical explanation for this discrepancy.

We then turn to a consideration of operators built out of arbitrary numbers of derivatives of any of the three scalar fields of Yang Mills theory. Already the cohomology ennumeration problem in this subsector appears to be a complicated combinatorial problem that we have not been able to solve in general, except for a simple case (see section 3.2) in which we use the combinatorics of syzygy [23, 24] to find the exact partition function. However we do present a rigorous upper bound on the growth with charge of this partition function and demonstrate that this growth is parametrically too slow (in the parameter N) to account for black hole entropy. Consequently, black hole entropy is presumably dominated by operators that are not Q equivalent to operators constructed out of scalars. We leave the study of such operators to future work.

Finally, in sections 4 and 5 we attempt to obtain a physical interpretation of the local generating fields that naturally appear in our description of the action of Q on 1/16 BPS letters. For this purpose it is useful to take a step backwards from the quantum problem we have studied so far, and study the manifold of classical 1/16 BPS configurations of $\mathcal{N}=4$ Yang Mills theory on $S^3 \times \mathbb{R}$ (or equivalently on R^4 with appropriate reality conditions) focusing on the bosonic sector. We will now discuss this in some detail.

The bosonic subgroup of the $\mathcal{N}=4$ superconformal algebra is $\mathrm{SO}(4,2)\times\mathrm{SO}(6)$. A basis for the Cartan subalgebra of this algebra is given by the energy E of states (the quantum number under the timelike SO(2) factor of SO(4,2)), the half integer J and $\bar{J}$ values of $\mathrm{SU}(2)\times\mathrm{SU}(2)\sim\mathrm{SO}(4)\in\mathrm{SO}(4,2)$ and H_1, H_2, H_3, the generators of rotations in orthogonal two planes in SO(6).[1] Let Q denote the supersymmetry operator of $\mathcal{N}=4$ Yang Mills, whose Cartan charges are given by $E=\frac{1}{2}$, $J=-\frac{1}{2}$, $\bar{J}=0$, $H_1=H_2=H_3=\frac{1}{2}$. It follows immediately from the superconformal algebra that states that are annihilated by both Q and its hermitian conjugate obey the BPS bound $\Delta\equiv E-2J-H_1-H_2-H_3=0$. We prove a classical version of this bound; in particular we demonstrate that the classical Noether charge corresponding to Δ is equal to a sum of squares, and so is positive definite. Classical bosonic configurations with $\Delta=0$ obey a set of first order Bogomolnyi equations obtained by setting each of these squares to zero.

While the first order supersymmetric equations are explicit, they are nonlinear and we do not know how to solve them in general. However these equations may be solved very simply when g_{YM} is set to zero, when the equations linearize. It turns out that the supersymmetric solutions of the free theory are parameterized by holomorphic data on the base $C^2=R^4$. The holomorphic data that parameterizes free solutions may, infact, be identified with the holomorphic generating functions on which the action of Q was conveniently defined, as described above.

It is then natural to ask whether each supersymmetric solution at $g_{\mathrm{YM}}=0$ admits a small perturbation to a supersymmetric solution at infinitesimal g_{YM}. This question may be addressed perturbatively and the answer is no. It turns out that supersymmetric solutions of the free theory must satisfy an infinite class of integrability constraints (the first of which is the integrated Gauss Law) in order that they may be perturbed to supersymmetric solutions at infinitesimal coupling.

Now the quantization of all supersymmetric states of free Yang Mills theory simply yields the Fock Space of supersymmetric Yang Mills 'letters'. Suitably interpreting the constraint equations above as quantum constraints that must be additionally imposed on Hilbert space, we recover a description of the supersymmetric Hilbert space that reduces to the description of Q cohomology described in the first part of this paper, but this time with a physical interpretation for the 'generating function' fields as physical Yang Mills fields propagating on physical R^4 but restricted to the supersymmetric sector.

The plan of the rest of this paper is as follows. In section 2 we state our cohomology problem, discuss the superposition of BPS states with low energy, review the classical cohomology problem and reformulate it using fields in terms of which the action of Q is local. We show in particular that the constraints of Q-cohomology for operators constructed purely from bosonic letters reduce to holomorphic, local gauge invariance and symmetrization of these fields. In section 3, we consider counting the states of the classical cohomology

[1] We shall also use J_1, J_2 rotating two orthogonal 2-planes, $J=\frac{J_1+J_2}{2}$ and $\bar{J}=\frac{J_1-J_2}{2}$.

which are made of scalar letters. We obtain exact partition functions in the single scalar sector and in the two scalar sector with U(2) gauge group, and also present an upper bound partition function for the general scalar cohomology. Using these results, we discuss the giant graviton interpretations and also show that the degeneracy of these cohomologies grows too slowly to form a black hole. In section 4 we present a set of bosonic classical $\frac{1}{16}$-BPS equations for $\mathcal{N} = 4$ Yang-Mills theory. In section 5 we focus on the solutions of these equations in weakly-coupled Yang-Mills theory. We show that the solutions of free theory should obey infinitely many constraints to be lifted to nearby solutions in the interacting theory. In section 6, we attempt to radially quantize the system. Imposing a natural quantization prescription in the weakly-coupled theory, we obtain a final condition which is equivalent to that of the Q-cohomology. We also obtain an interpretation of the local fields as physical Yang-Mills fields on $\mathbb{C}^2$. Appendix A reviews the supersymmetry transformations. Appendix B provides an independent check for our SU(2) two-scalar partition function in sectors with a few derivatives. Appendices C, D, E, F treat the details of sections 4 and 5 as well as presenting some exact solutions of our BPS equation.

2. The 1/16 BPS cohomology

$\mathcal{N} = 4$ Super-Yang-Mills on $S^3 \times \mathbb{R}$ has a $P\,\mathrm{SU}(2,2|4)$ group of global symmetries. The fermionic generators of this algebra are 16 supersymmetries, $Q^i_\alpha, \bar{Q}_{i\dot\alpha}$ and 16 super conformal generators $S^\alpha_i, \bar{S}^{i\dot\alpha}$. These generators transform in the $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)_R$ bosonic subgroup and an upper SU(4) index $i = 1 \ldots 4$ indicates a fundamental representation, while lower indices are antifundamental. With radial quantization, these generators satisfy the relations:

$$\begin{aligned} S^\alpha_i &= (Q^i_\alpha)^\dagger \quad & \bar{S}^{i\dot\alpha} &= (\bar{Q}_{i\dot\alpha})^\dagger \\ \bar{Q}_{i\dot\alpha} &= (Q^i_\alpha)^* \quad & \bar{S}^{i\dot\alpha} &= (S^\alpha_i)^* \end{aligned} \tag{2.1}$$

where $*$ denotes complex conjugation and $\dagger$ denotes hermitian conjugation.

We will be interested in states of the quantum theory which are annihilated by the minimum number of supercharges. These are 1/16 BPS states, which are annihilated by only a single supersymmetry, say Q^4_- (written as Q in the introduction), and its hermitian conjugate S^-_4. That is, we are interested in states that satisfy $Q^4_-|\psi\rangle = S^-_4|\psi\rangle = 0$. It will be convenient for us to adopt a slightly formal description of these 1/16 BPS states: If we formally regard Q^4_- as an exterior derivative d and S^-_4 as its Hermitian conjugate $d*$, then $\{Q^4_-, S^-_4\}$ corresponds to the Laplacian $\Delta = d * d + dd*$. Standard arguments show that states with $\Delta = 0$, which are harmonic forms, are in one-to-one correspondence with states in the cohomology of d. Analogous arguments, formulated in terms of Q^4_-, S^-_4 show that states which satisfy $\{Q^4_-, S^-_4\}|\Psi\rangle = 0$ are in one-to-one correspondence with states in the cohomology of Q^4_-.

Therefore, from now on, we will consider the set of 1/16 BPS states to be either all states that are annihilated by both Q^4_- and S^-_4, or all states that are Q^4_- closed but not Q^4_- exact. From the point of view of calculating a partition function over the 1/16 BPS states, the two formulations are equivalent.

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2.1 Cohomology at energies less than N

We will first consider the BPS cohomology of $\mathcal{N} = 4$ at energies less than N where we can characterize it fully.

Consider any basis for single trace operators in Yang Mills theory. The key observation here is that the action of Q on Yang Mills fields takes the form $QA = [B, C]$ where each of A, B, C are adjoint Yang — Mills letters (see appendix A). It follows that the action of Q on a single trace operator once again returns a single trace operator.[2] Consequently it is possible to choose a graded basis in the space of all single trace operators. Let $\{\gamma_i\}$ represent any basis of single trace operators such that $[Q, \gamma_i] \neq 0$, and let $\{\gamma_i, Q\gamma_i, \alpha_i\}$ represent a basis of all operators in the theory. As α_i are linearly independent of all states that are not Q closed ($\{\gamma_i\}$) and all states that are Q exact $\{Q\gamma_i\}$, the set $\{\alpha_i\}$ is a basis for Q cohomology in the single trace sector. We denote the linear space spanned by $\{\alpha_i\}$ by $\mathcal{H}^{\rm ST}_{\rm SUSY}$.

We will now argue that at energies (or scaling dimensions) less than N, the full BPS cohomology of Yang Mills theory is given by $\mathcal{F}(\mathcal{H}^{\rm ST}_{\rm SUSY})$. This result follows immediately from the observations that

1. The fock space of single trace operators constitutes basis for the space of all gauge invariant operators at scaling dimensions less than N.

2. The action of Q preserves the number of traces, and moreover Q acts in a trace by trace manner; for example

$$[Q, AB] = [Q, A]B \pm A[Q, B] = 0\,, \tag{2.2}$$

   where A and B are single trace operators.

3. The mathematical result that the fock space cohomology of an operator Q is the fock space of its cohomology (see for instance the discussion surrounding equation 12.4.23 of [25]).

It may be useful to illustrate point 3 above in two trace sector. Let A and B each belong to single trace cohomology. It is obvious that AB is Q closed. Further, it is always possible to choose our single trace basis such that A and B are α_1 and α_2 (provided A and B are not proportional). It is then clear that $AB = [Q, O]$ as any nonzero term on the r.h.s. of this equation contains a piece proportional to $[Q, \gamma_i]$ for some γ_i and so cannot equal $\alpha_1\alpha_2$. Consequently AB belongs to Q cohomology. Further if A and A' are Q equivalent it is obvious that AB and $A'B$ are Q equivalent for any Q closed B. Finally it is also clear that the operators $\alpha_i\alpha_j$ are Q inequivalent for different values of the pair of indices $\{i, j\}$. All this establishes that the symmetric product of single trace cohomology lies within the fock space of Q acting on the symmetric product Hilbert space. Similar arguments may be used to establish the strict equality of these two constructions.

[2]At high energies it may be possible to use trace identities to rewrite this operator as a polynomial in multi traces, but one may always choose not to do so. At any event we eventually focus, in this section, on energies less than N where such a rewriting is impossible.

We now turn the question: What is the cohomology of Q in the single trace sector? While we are not aware of a complete proof of this result, it seems overwhelmingly likely that this cohomology is simply given by the set of 1/16 BPS descendants of chiral primary operators, i.e. the list of 1/16 BPS single gravitons in AdS_5. Nontrivial evidence for this conjecture was reported in [22]. Assuming this to be the case, we have proved in this section that the cohomology of Q in is given by the fock space of 1/16 BPS gravitons at energies smaller than N. As we have discussed in the introduction this is already a nontrivial result; it implies that $1/N$ effects cannot renormalize the energy of a collection of 1/16 BPS gravitons.

At energies larger than N the arguments of this subsection no longer apply; indeed that is a good thing as the entropy of the fock space of supersymmetric gravitons grows with energy like $E^{\frac{5}{6}}$, a growth that is too slow to account for the $\mathcal{O}(N^2)$ states at energies of order N^2 of 1/16 BPS supersymmetric black holes. In the next section we will begin a systematic investigation of Q cohomology at energies larger than N. Unfortunately we will be able to report only modest progress in characterizing this cohomology.

2.2 Generalities on classical cohomology

$\mathcal{N}=4$ SYM has 6 scalars Φ_{ij}, 4 chiral fermions $\Psi_{i\alpha}$ and a gauge field $A_{\alpha\dot{\beta}}$. The scalars Φ_{ij} are in the antisymmetric product of SU(4). The scalar fields satisfy $\Phi^*_{ij}=\Phi^{ij}$ where $\Phi^{ij}=\frac{1}{2}\epsilon^{ijkl}\Phi_{kl}$. The complex conjugates of the fermions are $\bar{\Psi}^i_{\dot{\alpha}}$.

We will write the field content in terms of chiral fields defined as (we use the convention $\epsilon^{4mnp}=+\epsilon^{mnp}$):

$$\Phi^{4m}=\bar{\phi}^m \qquad \frac{1}{2}\epsilon_{pmn}\Phi^{mn}=\Phi_{4p}=\phi_p \qquad \Psi_{4\alpha}=\lambda_\alpha \qquad \Psi_{m\alpha}=\psi_{m\alpha} \tag{2.3}$$

with $m,n,p=1\dots 3$.

Also, from now on we will denote the special supercharges as $Q^4_\alpha=Q_\alpha$, $S^\alpha_4=S^\alpha$ and denote the remaining supercharges as Q^m_α with $m=1,2,3$. With these definitions, the action of the supercharges on the fields is listed in appendix A.

To begin with, we consider the cohomology of Q_- at zero coupling, where all commutators in the supersymmetry algebra vanish. The supersymmetry algebra has:

$$\Delta \equiv 2\{Q_-,S^-\}=E-2J-H_1-H_2-H_3 \tag{2.4}$$

where E is the dilatation operator, or the energy in radial quantization, J is the left SU(2) charge and H_i are the SU(4) Cartans. At zero coupling, we can solve the cohomology problem by simply listing all basic fields or 'letters' in the theory which have $\Delta=0$. These are $\bar{\phi}^m$, $\bar{\lambda}_{\dot{\alpha}}$, ψ_{m+}, f_{++} and derivatives $D_{+\dot{\alpha}}$ acting on them, where f_{++} denotes the ++ component of the field strength $f_{\alpha\beta}$. A quick look at the supersymmetry algebra shows that these letters are Q_- closed, but not Q_- exact. The gaugino equation of motion,

$$\partial_{+\dot{\beta}}\bar{\lambda}^{\dot{\beta}}=0\,, \tag{2.5}$$

is the only equation of motion that can be constructed out of these letters. At zero coupling, any operator constructed out of the $\Delta=0$ letters, modulo this equation of motion, will be 1/16 BPS. The partition function over the above letters can easily be calculated and can be found in [8].

At finite coupling, the commutators appearing on the right hand side of the supersymmetry algebra in appendix A introduce constraints on the free cohomology. The essential point in what follows is that we will formulate the supersymmetry algebra in a way that makes some of these constraints easy to implement in some sectors.

The action of the special supercharge Q_- on the supersymmetric letters is:

$$\begin{aligned}
\left[Q_-,\bar\phi^n\right] &= 0\\
\{Q_-,\psi_{n+}\} &= \epsilon_{nmp}\left[\bar\phi^m,\bar\phi^p\right]\\
\{Q_-,\bar\lambda_{\dot\beta}\} &= 0\\
[Q_-,f_{++}] &= i\left[\bar\phi^m,\psi_{m+}\right]\ .
\end{aligned}\tag{2.6}$$

We define a field corresponding to each supersymmetric letter to simplify the analysis of derivatives:

$$\begin{aligned}
\bar\phi^m(z) &= \sum_n \frac{z_1^{n_1}z_2^{n_2}}{n_1!n_2!}D_1^{n_1}D_2^{n_2}\bar\phi^m\\
f(z) &= \sum_n \frac{z_1^{n_1}z_2^{n_2}}{n_1!n_2!}D_1^{n_1}D_2^{n_2}f_{++}\\
\psi_{m+}(z) &= \sum_n \frac{z_1^{n_1}z_2^{n_2}}{n_1!n_2!}D_1^{n_1}D_2^{n_2}\psi_{m+}\\
\bar\lambda_{\dot\alpha}(z) &= \sum_n \frac{z_1^{n_1}z_2^{n_2}}{n_1!n_2!}D_1^{n_1}D_2^{n_2}\bar\lambda_{\dot\alpha}\ .
\end{aligned}\tag{2.7}$$

The derivatives $D_{+\dot\alpha}$ have been abbreviated as D_1, D_2 and all derivatives should be understood to be symmetrized. That is $D_1D_2\phi$ denotes $\frac{1}{2}(D_1D_2+D_2D_1)\phi$ and so on. With these definitions the action of Q_- on the supersymmetric derivatives is

$$[Q_-,D_{\dot\alpha}\zeta] = -i[\bar\lambda_{\dot\alpha},\zeta]+D_{\dot\alpha}Q_-\zeta\,,\tag{2.8}$$

and the action of Q_- on the supersymmetric fields is

$$\begin{aligned}
[Q_-,\bar\phi^m(z)] &= -i[z^{\dot\alpha}(1+z\cdot\partial)^{-1}\bar\lambda_{\dot\alpha}(z),\bar\phi^m(z)]\\
[Q_-,f(z)] &= -i[z^{\dot\alpha}(1+z\cdot\partial)^{-1}\bar\lambda_{\dot\alpha}(z),f(z)]+i\ [\bar\phi^n(z),\psi_{n+}(z)]\\
\{Q_-,\bar\lambda_{\dot\beta}(z)\} &= -i\{z^{\dot\alpha}(1+z\cdot\partial)^{-1}\bar\lambda_{\dot\alpha}(z),\bar\lambda_{\dot\beta}(z)\}\\
\{Q_-,\psi_{m+}(z)\} &= -i[z^{\dot\alpha}(1+z\cdot\partial)^{-1}\bar\lambda_{\dot\alpha}(z),\psi_{m+}(z)]+\epsilon_{mnp}[\bar\phi^n(z),\bar\phi^p(z)]\ .
\end{aligned}\tag{2.9}$$

In particular, the action of Q_- has two terms: The first has the form of an infinitesimal gauge transformation parameterized by the object $z^{\dot\alpha}(1+z\cdot\partial)^{-1}\bar\lambda_{\dot\alpha}(z)$ and the second is completely local with these field definitions.

We include here a proof of the above supersymmetry transformations for fields which are independent of z_2. For simplicity of notation, we write $Q_- = Q$ and $D_1 = D$:

$$\begin{aligned}
[Q,\xi(z)] &= z\sum_n \left(\frac{z^{n-1}}{n!}\sum_{k=0}^{n-1}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}\left[D^l\lambda, D^{n-l-1}\xi\right]\right) + \sum_n \frac{z^n}{n!}D^n[Q,\xi] \\
&= -iz\sum_n \left(\frac{z^{n-1}}{n!}\sum_{l=0}^{n-1}\left(\sum_{k=l}^{n-1}\frac{k!}{l!(k-l)!}\right)\left[D^l\lambda, D^{n-l-1}\xi\right]\right) + \sum_n \frac{z^n}{n!}D^n[Q,\xi] \\
&= -iz\sum_n \left(\frac{z^{n-1}}{n!}\sum_{l=0}^{n-1}\frac{n!}{(l+1)!(n-l-1)!}\left[D^l\lambda, D^{n-l-1}\xi\right]\right) + \sum_n \frac{z^n}{n!}D^n[Q,\xi] \\
&= -iz\sum_n\sum_{l=0}^{n-1}\left[\frac{D^l\lambda}{(l+1)!}z^l, \frac{D^{n-l-1}\xi}{(n-l-1)!}z^{n-l-1}\right] \\
&= -i\left[z(1+z\partial)^{-1}\lambda(z),\xi(z)\right] + [Q,\xi](z)
\end{aligned} \tag{2.10}$$

The general case, allowing both derivatives follows by similar arguments which keep track of the fact that derivatives are symmetrized. We note that it is also possible to write the action of some of the supercharges which commute with Q_-, S^- in a form similar to that in equation 2.9.

2.3 1/16 BPS cohomology at finite N in the sector made of bosonic operators

In this subsection, we will describe the 1/16 BPS cohomology in the sector where operators are constructed from any number of derivatives, scalars and gauge fields. More precisely we will study the counting all elements of Q cohomology that are Q equivalent to a purely bosonic operator.

Let us first study consequences of the requirement that the operators we study are Q closed. Recall that the action of Q on $\bar{\phi}^m$ is proportional to the gaugino operators. Since we are considering only operators which themeselves contain no gauginos, we may simply regard the gaugino field as an arbitrary holomorphic fermionic field. Consequently, the transformation

$$Q_-\bar{\phi}^m(z) = -i[z^{\dot{\alpha}}(1+z\cdot\partial)^{-1}\bar{\lambda}_{\dot{\alpha}}(z),\bar{\phi}^m(z)] \tag{2.11}$$

is simply a gauge transformation parameterized by $W(z_1,z_2) = z^{\dot{\alpha}}(1+z\cdot\partial)^{-1}\bar{\lambda}_{\dot{\alpha}}(z)$. We conclude that the set of Q invariant operators constructed out of the fields $\bar{\phi}^m(z)$ is simply the set of operators made out of these fields that are invariant under z dependent $\mathrm{U}(N)$ gauge transformations acting on $\bar{\phi}^m(z)$.

We reiterate that Q closed operators constructed out of $\bar{\phi}^m(z)$ (and, as we will see below the gauge field $f(z)$) must be gauge invariant under all *holomorphic* gauge transformations $W(z_1,z_2)$. If we choose to construct gauge invariants using traces, the requirement of Q closedness requires that every field inside any given trace is evaluated at the same z. Of course different traces may be evaluated at different values of z. For example, if $x \neq y$, we may consider $\mathrm{tr}\,\bar{\phi}^1(x)\bar{\phi}^2(x)$, but not $\mathrm{tr}\,\bar{\phi}^1(x)\bar{\phi}^2(y)$.

In order to understand the constraints from Q exactness consider

$$\{Q_-,\psi_{m+}(z)\} = -i\left\{z^{\dot{\alpha}}(1+z\cdot\partial)^{-1}\bar{\lambda}_{\dot{\alpha}}(z),\psi_{m+}(z)\right\} + \epsilon_{mnp}\left[\bar{\phi}^n(z),\bar{\phi}^p(z)\right]. \tag{2.12}$$

This relation implies that operators containing commutators of scalars inside a trace are Q_- exact. In particular if $|\chi\rangle = \mathrm{tr}\left(A(z)[\bar\phi^m(z), \bar\phi^n(z)]\right) \times \ldots$ and $|\chi\rangle' = \frac{1}{2}\,\mathrm{tr}\left(A(z)\epsilon^{mnk}\psi_{k+}(z)\right)$ then $Q_-|\chi\rangle' = |\chi\rangle$ (the terms in Q variation that involve the gaugino cancel out because of the 'gauge' invariance of $|\chi\rangle'$ as described in the previous paragraph).

Now let us study the requirement of Q invariance of operators containing the gauge field f. We see that the two terms on the right hand side of

$$[Q_-, f(z)] = -i\left[z^{\dot\alpha}(1+z\cdot\partial)^{-1}\bar\lambda_{\dot\alpha}(z), f(z)\right] + i\ \left[\bar\phi^n(z), \psi_{n+}(z)\right] \tag{2.13}$$

must be annihilated separately since one involves a gaugino and the other a chiralino. The first term is the same gauge transformation we saw above in equation (2.11) and the second constraint ensures that no operator constructed purely from bosonic letters may contain $f(z)$ except the operator $\mathrm{tr}\, f(z)$ in the $\mathrm{U}(N)$ theory.

Therefore states in Q cohomology in the $\mathrm{SU}(N)$ theory that are composed entirely out of bosonic letters can chosen to satisfy two constraints: They must be gauge invariant functions of the local fields $\bar\phi^m(z)$ and they can be chosen to be completely symmetrized on all scalars inside any given trace.

We have so far been studying operators constructed out of the generating functions $\bar\phi^m(z)$. While these generating fields are convenient for many purposes, they do not carry definite values of the angular momentum quantum number J. If we are interested in counting operators graded by J, as we typically are, then we must eventually return to the derivative basis. In this basis the single trace operators in Q cohomology are

$$D_1^{k_1} D_2^{k_2}\,\mathrm{tr}\left(\bar\phi^1\right)^{n_1}\left(\bar\phi^2\right)^{n_2}\left(\bar\phi^3\right)^{n_3}\,, \tag{2.14}$$

where D_1, D_2 refer to the derivatives $D_{+\dot\alpha}$, which can be regarded as ordinary derivatives since they act on gauge-invariants. The scalars inside the trace is regarded as being symmetrized. The only remaining constraints on the cohomology in this sector are the trace relations which become important for operators with more than N letters and reduce the number of independent operators. In the $\mathrm{U}(N)$ theory, the field strength $f(z)$ may also participate, but in a rather trivial way as explained above.

This description of the scalar operators in Q cohomology is not yet explicit enough to provide a simple counting rule to ennumerate these operators. The reason for this is that we have not yet come to grips with the trace identities that complicate this ennumeration. We will have only modest success in taming these identities in the next section.

3. Partition functions at finite N

In this section, we will compute the exact partition function of the $\frac{1}{16}$-BPS cohomology in several subsectors involving scalars and derivative operators only: the latter restriction meaning that the cohomology has at least one representative made of scalars and derivatives only, as discussed in the previous section.

To be concrete, we provide a partition function of cohomologies involving one species of scalar and arbitrary derivatives in $\mathrm{U}(N)$ gauge theory, and also involving two species of scalars and arbitrary derivatives in U(2) gauge theory. The combinatorics of plethystics and syzygies, explored recently in [23, 24] in the context of chiral rings, proves useful in obtaining the partition function of the two scalar species U(2) subsector.

We also investigate a partition function which provides an upper bound for the exact degeneracy of $\frac{1}{16}$-BPS cohomology involving all scalars and derivatives. The latter upper bound is obtained by loosening the condition for the $\frac{1}{16}$-BPS cohomology and we explain that it has a clear interpretation in the dual gravity context. The same upper bound partition function can be obtained by 'naively' quantizing the $\frac{1}{16}$-BPS fluctuations on dual giant gravitons in $AdS_5 \times S^5$. The purpose of studying the upper bound partition function is twofold. One is to conclusively show that one cannot reproduce the entropy of supersymmetric black holes from the purely bosonic cohomology. Another purpose is to try to have a better understanding of the 1/16 BPS states in the regime $g_s N \gg 1$, $E \gtrsim N$ using giant gravitons.

3.1 Partition function of a single scalar and all derivatives at finite N

Now we will consider the sector generated by a single scalar, say $\bar{\phi}^1$ and all derivatives. Our prescription states that the operators[3] we should count are generated as products of the single trace states $\mathrm{tr}\left(\bar{\phi}^1(z)\right)^n$. These operators may be counted very simply at a given z even accounting for trace identities; the answer is given simply by all polynomials of $\mathrm{tr}\left(\bar{\phi}^1(z)\right)^n$ [26]. In order to account for the z dependence we must count all polynomials of

$$D_1^{k_1} D_2^{k_2}\,\mathrm{tr}\left(\bar{\phi}^1\right)^n \qquad n = 1, \ldots, N \tag{3.1}$$

The multi-trace partition function is given by the formulae of Bose statistics

$$Z_N(\mu_1, \theta_1, \theta_2) = \prod_{k_1,k_2=0}^{\infty} \prod_{n=1}^{N} \frac{1}{1 - \theta_1^{k_1} \theta_2^{k_2} \mu_1^n} \tag{3.2}$$

We now turn to the bulk interpretation of this partition function. Let us first study our partition function in terms of giant gravitons. The giant gravitons which are dual to the operators considered in this section are 1/8 BPS D3-branes that wrap S^3s with a particular orientation on the S^5 and move on a given (pointlike) trajectory in AdS_5 [27, 28, 21]. The set of such configurations can be quantized and the resulting partition function was computed in [21] and agrees exactly with 3.2.

It is natural to inquire whether this partition function admits other complementary bulk interpretations. Recall that the BPS sector of Yang-Mills chiral ring admited two complementary quantizations; the first [29, 20] from quantizing gravitons [30], and the second quantizing by quantizing dual giant gravitons [21]. (The latter approach has been generalized [31, 32] to certain $\mathcal{N} = 1$ superconformal theories.) The two descriptions yield exactly the same partition function in this sector, which also agrees with the Yang-Mills

[3]Such states are in fact 1/8 BPS states, but preserve a different set of supercharges from the better studied chiral ring. The chiral ring (which can be constructed from the zero modes of the 3 complex scalars, $\bar{\phi}^i$) counts states that are annihilated by Q_α and S^α, but the states of this subsection are annihilated by Q_-, S^- and Q^1_-, S^-_1 instead.

theory result [8]. In section 3.4 we will discuss problems with an analogeous dual giant interpretation of (3.2)

3.2 Exact partition function in sector with 2 scalars for $\mathrm{U}(2)$

In the last subsection we presented an exact partition function over operators in Q cohomology that are composed of arbitrary numbers of derivatives of a single scalar field. In this subsection we will present the only other exact partition function in our paper - the partition function over operators in Q cohomology that are composed of arbitrary numbers of derivatives of two scalar fields - but only for the gauge group $\mathrm{U}(2)$.

Restated, in this section, we will calculate explicitly the partition function in the sector with 2 scalars, and all derivatives for gauge group $\mathrm{U}(2)$. For convenience, we denote the 2 scalars as X, Y. We will employ the plethystic exponential described in [23, 24]. An independent check of our partition function can be found in appendix B.

We first consider multi trace states constructed from the two scalars X, Y without any derivatives. These are 1/4 BPS states and their counting is well known; see for example [8]. At finite N, one must restrict the total number of traces used to form independent multi-trace operators to be less or equal to N. For convenience, we include a trivial identity operator $\frac{1}{N}\mathrm{tr}(\mathbf{1}_N)$ and constrain the number of traces to be exactly N. The partition function over the 1/4 BPS states in the $\mathrm{U}(N)$ theory is the coefficient of p^N in

$$Z(p, \mu_1, \mu_2) \equiv \sum_{N=0}^{\infty} p^N Z_N(\mu_1, \mu_2) = \prod_{n_1, n_2=0}^{\infty} \frac{1}{1 - p\ \mu_1^{n_1} \mu_2^{n_2}} \ . \tag{3.3}$$

The variable p is a chemical potential for the number of traces. In particular, the $\frac{1}{4}$-BPS partition function for the $\mathrm{U}(2)$ theory can be can be obtained by computing $Z_2 = \frac{1}{2}\frac{\partial^2}{\partial p^2} Z(p, \mu_1, \mu_2)\Big|_{p=0}$:

$$Z_2(\mu_1, \mu_2) = \frac{1 + \mu_1 \mu_2}{(1-\mu_1)^2 (1-\mu_2)^2 (1+\mu_1)(1+\mu_2)} \ . \tag{3.4}$$

An alternative way of imposing the trace relation constraint is to leave the number of traces unrestricted, but to instead restrict the number of letters inside each trace, as discussed in section 3.1. For gauge group $\mathrm{U}(2)$, the following single trace operators generate the most general multi-trace operators in the 1/4 BPS sector:

$$\mathcal{O}_{n_1 n_2} \equiv \mathrm{tr}\left(X^{n_1} Y^{n_2}\right) \ , \quad n_1 + n_2 \leq 2 \, , \tag{3.5}$$

They are called primitive operators, or generators. While these primitive operators generate all multi-trace operators in the 1/4 BPS sector, there is a redundancy, or overcounting of operators since there may be more than one polynomial relation among $\mathcal{O}_{n_1 n_2}$ arising from trace relations. Such relations are called *syzygies* [24]. Explicitly, in the $\frac{1}{4}$-BPS sector with $\mathrm{U}(2)$ gauge group, there are five primitive operators

$$\mathcal{O}_{10} = \mathrm{tr}(X) \, , \quad \mathcal{O}_{01} = \mathrm{tr}(Y) \, , \quad \mathcal{O}_{20} = \mathrm{tr}(X^2) \, , \quad \mathcal{O}_{11} = \mathrm{tr}(XY) \, , \quad \mathcal{O}_{02} = \mathrm{tr}(Y^2) \, , \tag{3.6}$$

which are subject to only one syzygy (see section 6 of [24])

$$\mathcal{O}_{20}(\mathcal{O}_{01})^2 + \mathcal{O}_{02}(\mathcal{O}_{10})^2 + 2(\mathcal{O}_{11})^2 - 2\mathcal{O}_{20}\mathcal{O}_{02} - 2\mathcal{O}_{10}\mathcal{O}_{01}\mathcal{O}_{11} = 0 \; . \tag{3.7}$$

One may regard this syzygy as relating $(\mathcal{O}_{11})^2 = (\mathrm{tr}(XY))^2$ to a combination of primitive operators containing no more than one $\mathcal{O}_{11}$.

The partition function of the primitive single trace operators is

$$z_2'(\mu_1, \mu_2) \equiv \mu_1 + \mu_2 + \mu_1^2 + \mu_1\mu_2 + \mu_2^2 \; . \tag{3.8}$$

Had one been ignoring the syzygy, we would have obtained a multi-trace partition function simply by multiparticling (or taking plethystic exponential of) this single trace partition function

$$Z_2'(\mu_1, \mu_2) = \exp\left(\sum_{r=1}^{\infty} \frac{z_2'(\mu_1^r, \mu_2^r)}{r}\right) = \frac{1}{(1-\mu_1)(1-\mu_2)(1-\mu_1^2)(1-\mu_2^2)(1-\mu_1\mu_2)} \; . \tag{3.9}$$

Comparing with the correct answer $Z_2(\mu_1, \mu_2)$, we find that Z_2' overcounts the states, simply because to the relation (3.7) is ignored. subtracting this, the correct partition function should be

$$Z_2(\mu_1, \mu_2) = Z_2'(\mu_1, \mu_2) - \mu_1^2\mu_2^2 Z_2'(\mu_1, \mu_2) = \left(1 - \mu_1^2\mu_2^2\right) Z_2'(\mu_1, \mu_2) \, , \tag{3.10}$$

which is indeed true. The second subtracted term corresponds to eliminating the contribution to the partition function from operators of the form

$$\Big(\mathrm{tr}(XY)\Big)^2 \Big(\text{arbitrary multiplication}\Big) \, , \tag{3.11}$$

since we do not want to count any operators containing more than one $\mathcal{O}_{11} = \mathrm{tr}(XY)$.

The above elimination of overcounting, or compensation for syzygies, can be phrased in terms of the single trace partition function. The factor $(1-\mu_1^2\mu_2^2)$ in (3.10) can be regarded as coming from the Plethystic exponential of $-\mu_1^2\mu_2^2$. In other words, the 'effective' single trace partition function which gives the correct multi-trace answer is

$$z_2(\mu_1, \mu_2) = \mu_1 + \mu_2 + \mu_1^2 + \mu_1\mu_2 + \mu_2^2 - \mu_1^2\mu_2^2 \; . \tag{3.12}$$

Formally, the last term may be regarded as eliminating the redundant generator $(\mathrm{tr}(XY))^2$ by giving it 'degeneracy' -1.

Now we wish to add derivatives to each single trace operator. First, to each term in the single-trace partition function with positive coefficients (corresponding to a primitive operator), we multiply a factor

$$\frac{1}{(1-\theta_1)(1-\theta_2)} \equiv \sum_{k_1,k_2=0}^{\infty} (\theta_1)^{k_1}(\theta_2)^{k_2} \, , \tag{3.13}$$

where θ_1 and θ_2 are chemical potentials for the two angular momenta, i.e., numbers of derivatives. Furthermore, the form of states that should be eliminated (due to syzygies) should also be multiplied by this factor, since any set of derivatives acting on a syzygy also represents a redundancy to be eliminated from the counting. Therefore, one obtains the following single-particle and multi-particle partition functions:

$$z_2(\mu_1,\mu_2,\theta_1,\theta_2) = \frac{z_2(\mu_1,\mu_2)}{(1-\theta_1)(1-\theta_2)} \tag{3.14}$$

$$Z_2(\mu_1,\mu_2,\theta_1,\theta_2) = \exp\left(\sum_{r=1}^{\infty}\frac{z_2(\mu_1^r,\mu_2^r,\theta_1^r,\theta_2^r)}{r}\right) . \tag{3.15}$$

Following the above procedure, one finds

$$\begin{aligned} &Z_2(\mu_1,\mu_2,\theta_1,\theta_2) = \\ &= \prod_{k_1,k_2=0}^{\infty}\frac{1-\mu_1^2\mu_2^2\theta_1^{k_1}\theta_2^{k_2}}{\left(1-\mu_1\theta_1^{k_1}\theta_2^{k_2}\right)\left(1-\mu_2\theta_1^{k_1}\theta_2^{k_2}\right)\left(1-\mu_1^2\theta_1^{k_1}\theta_2^{k_2}\right)\left(1-\mu_2^2\theta_1^{k_1}\theta_2^{k_2}\right)\left(1-\mu_1\mu_2\theta_1^{k_1}\theta_2^{k_2}\right)} . \end{aligned} \tag{3.16}$$

In the appendix B we provide an independent nontrivial check of this result.

It is natural to wonder whether the U(N) partition function including all three scalars and all derivatives may be explicitly calculated by the same procedure using the single trace basis. With U(2) group, an explicit check of the kind carried out in the appendix B, which can be easily extended to the cases with three scalars, shows that this is *not* the case. We find that an apparent reason for this failure seems to be the following: while the single trace partition function (3.12) is a finite series with two scalars, the similar function becomes an infinite series with three scalars in U(2). This is pointed out to correspond to the fact that $\mathbb{C}^4/\mathbb{Z}_2$ is a 'complete intersection,' while $\mathbb{C}^6/\mathbb{Z}_2$ is not [24]. There could be more delicate combinatoric structures applicable to all cases, beyond what we found in the case with U(2) two scalars.

3.3 Upper bound on scalar sector and high energy scaling

While we have not been able to find an exact formula to count all operators in Q cohomology that admit representatives composed purely of scalar fields, in this subsection we will present a rigorous upper bound for the growth in the number of such operators as a function of their energy. In particular we will demonstrate that when the charges and energy of operators are taken to be $O(N^2)$, this sector does not contain enough operators to reproduce the N^2 scaling of the entropy of black holes in $AdS_5\times S^5$.

Firstly, for the purpose of enumeration, the scalars in our cohomology may be regarded as commutating, or diagonal, matrices. Our upper bound is obtained in the U(N) theory by allowing any number of derivatives to act on any of the N eigenvalues of any of the three scalars. That is, we will ignore the fact that the eigenvalues should really be symmetrized *inside* each trace, before the action of the derivatives. Counting this larger set of operators will give us an upper bound for the number of operators in the scalar and derivatives sector.

Let us first characterize the above 'relaxed' Hilbert space. This simply consists of the N bosonic particle states made of the single eigenvalue Hilbert space, which we call $\mathcal{H}_1$. The last $\mathcal{H}_1$ is characterized as follows: We have the letters

$$\partial_1^{k_1}\partial_2^{k_2}x_i \qquad k_j=0,\ldots,\infty,\ i=1,2,3 \tag{3.17}$$

and $\mathcal{H}_1$ is made of all words constructed from these commuting letters. Then the full partition function is the coefficient of p^N in

$$Z = \prod_{n_k^i=0}^{\infty} \frac{1}{1 - pe^{-\beta \sum_{k,i} n_k^i (1+k_1^i+k_2^i)}}\,, \tag{3.18}$$

where the $k_j^i = 0, \ldots, \infty$ and $i = 1, 2, 3$ and $j = 1, 2$. (We have set $e^{-\beta} = \mu_i = \theta_a$ for simplicity.)

Let us pause for a brief comment. As we have described above, the limit $N \to \infty$ our relaxed Hilbert space is given by the Fock space the Hilbert space of supersymmetric scalar states of U(1) Yang Mills theory. On the other hand the restriction to scalars of the correct $N \to \infty$ limit of the 1/16 BPS partition function (see section 2) is given by the Fock space of a much smaller Hilbert space - the space of scalar 1/16 BPS descendents of chiral primaries. This makes clear that our relaxed Hilbert space is much larger than the actual Hilbert space a energies small compared to N. On the other hand we argue in the next section that the relaxed Hilbert space is not very different from the exact space at the opposite high energy end.

Now one obtains the free energy and N in terms of p and the temperature β:

$$\begin{aligned} F &= -1/\beta \ln Z = \frac{1}{\beta} \sum_{n_k^i=0}^{\infty} \ln \left(1 - pe^{-\beta \sum_{k,i} n_k^i (1+k_1^i+k_2^i)}\right) \\ \langle N \rangle &= p\frac{\partial}{\partial p} \ln Z = \sum_{n_k^i=0}^{\infty} \frac{1}{p^{-1} e^{\beta \sum n_k^i (1+k_1^i+k_2^i)} - 1}\,. \end{aligned} \tag{3.19}$$

p must take a value in $[0, 1]$. We now consider the high temperature limit $\beta \ll 1$. In this limit, there are a large number of states such that $\beta \sum n_k^i (1 + k_1^i + k_2^i) \sim 0$ so that if $p \sim 1$, these states will produce a divergent contribution to the right hand side of (3.19). Since the left hand side is finite, it must be that $p \ll 1$ when $\beta \ll 1$.

In this case, we may approximate the particle number as

$$\begin{aligned} \langle N \rangle \approx \sum_{n_k^i=0}^{\infty} pe^{-\beta \sum n_k^i (1+k_1^i+k_2^i)} &= p \prod_{k_j^i} \sum_{n_k^i=0}^{\infty} e^{-\beta n_k^i (1+k_1^i+k_2^i)} \\ &= p \left[\prod_{k_1,k_2=0}^{\infty} \frac{1}{1 - e^{-\beta(1+k_1+k_2)}} \right]^3 . \end{aligned} \tag{3.20}$$

Next we calculate $\langle E \rangle$:

$$\begin{aligned}
\langle E \rangle &= \sum_{n_k^i=0}^{\infty} \frac{\sum n_k^i(1+k_1^i+k_2^i)}{p^{-1}e^{\beta \sum n_k^i(1+k_1^i+k_2^i)}-1} \approx p \sum_{n_k^i=0}^{\infty} e^{-\beta \sum n_k^i(1+k_1^i+k_2^i)} \left[\sum n_k^i(1+k_1^i+k_2^i)\right] \\
&= \left[\prod_{k_1,k_2=0}^{\infty} \frac{1}{1-e^{-\beta(1+k_1+k_2)}}\right]^3 \left\{3p\sum_k \frac{1+k_1+k_2}{e^{\beta(1+k_1+k_2)}-1}\right\} \\
&= 3N\left(\sum_k \frac{1+k_1+k_2}{e^{\beta(1+k_1+k_2)}-1}\right) . \qquad (3.21)
\end{aligned}$$

We will next extract the $\beta \to 0$ asymptotic form of the series

$$\sum_{k_1,k_2=0}^{\infty} \frac{1+k_1+k_2}{e^{\beta(1+k_1+k_2)}-1} . \qquad (3.22)$$

Defining $x \equiv \beta k_1$ and $y \equiv \beta k_2$, we find that dx and dy are small in the $\beta \to 0$ limit, so that we can approximate the series by a 2-dimensional integral. Ignoring 1 in the $1+k_1+k_2$ which only gives subleading terms in β, one obtains

$$\sum_{k_1,k_2=0}^{\infty} \frac{1+k_1+k_2}{e^{\beta(1+k_1+k_2)}-1} \approx \frac{1}{\beta^3}\int_0^{\infty} dx \int_0^{\infty} dy \, \frac{x+y}{e^{x+y}-1} . \qquad (3.23)$$

Defining $t \equiv x+y$ and $u \equiv \frac{x-y}{2}$, this term becomes

$$\frac{1}{\beta^3}\int_0^{\infty} dt \int_{-\frac{t}{2}}^{\frac{t}{2}} du \, \frac{t}{e^t-1} = \frac{1}{\beta^3}\int_0^{\infty} dt \frac{t^2}{e^t-1} = \frac{2\zeta(3)}{\beta^3} , \qquad (3.24)$$

where $\zeta(s) \equiv \sum_{k=1}^{\infty} \frac{1}{k^s}$ is the Riemann's zeta function. Note that $\zeta(3) = 1.2020569\ldots$

The above evaluation gives

$$\langle E \rangle = \frac{6N\zeta(3)}{\beta^3} . \qquad (3.25)$$

Next we process F, using the fact that $pe^{-\beta \sum_{k,i} n_k^i(1+k_1^i+k_2^i)} \ll 1$, which allows us to truncate the log series:

$$F \approx -\frac{p}{\beta}\sum_{n_k^i=0}^{\infty} e^{-\beta \sum_{k,i} n_k^i(1+k_1^i+k_2^i)} = -\frac{p}{\beta}\left[\prod_{k_1,k_2=0}^{\infty} \frac{1}{1-e^{-\beta(1+k_1+k_2)}}\right]^3 = -\frac{N}{\beta} . \qquad (3.26)$$

Since $S = \beta(E-F) \simeq \frac{6\zeta(3)N}{\beta^2} + N$, we have in the limit of small β,

$$S = \frac{6\zeta(3)N}{\beta^2}. \qquad (3.27)$$

Eliminating β from the expressions for $\langle E \rangle$ and S, we find that

$$S = \Big(6\zeta(3)N\Big)^{1/3} E^{2/3}. \qquad (3.28)$$

So for states with energy $\mathcal{O}(N^2)$, we have $S \sim N^{5/3} < N^2$. This establishes conclusively that there are not enough states in this sector to form a black hole.

Note the entropy of a fock space of scalar 1/16 descendants of chiral primaries grows like $KE^{5/6}$ where K is a number of order unity. Thus (3.28) has more states than multi-gravitons for $E \ll N^2$ (this is because relaxation of the Hilbert space greatly increases the number of states at low energies) but fewer states than multigravitons at $E \gg N^2$ (because (3.28) accounts for finite N truncations absent in the Fock space, which are important at high energies). The two ennumerations yield approximately the same number of states at energies of order N^2.

3.4 Upper bound, exact degeneracy and (dual) giant gravitons

In the $\frac{1}{8}$-BPS sector, the degeneracy of chiral primary operators has been convincingly reproduced from the bulk perspective by quantizing giant gravitons [30, 29, 20, 21]. Two complementary approaches are available; one which quantizes the giant gravitons extended in S^5, and another which quantizes the dual giant gravitons extended in AdS_5. In the $\frac{1}{16}$-BPS sector, or more generally in a sector with nonzero angular momentum in AdS_5, such an understanding from the bulk viewpoint is almost lacking, except in the very simple sector investigated from the bulk viewpoint in [21] and from gauge theory in our section 3.1.

The upper bound we provided in the previous subsection has an interpretation from the bulk perspective. We will show that it can be regarded as the result of a naive quantization of the $\frac{1}{16}$-BPS fluctuations [33] on the dual giant gravitons of [21]. As we have already seen in the previous subsection, the upper bound relaxation is far from a good approximation of the correct 1/16 BPS Hilbert space at low energies. In this section we will explain why (from a dual bulk perspective) the naive dual graviton quantization fails, and also explain when we expect it to yield approximately reliable results.

First, we briefly review the $\frac{1}{16}$-BPS fluctuations on dual giant gravitons and interpret the upper bound partition function in terms of such fluctuations. The giant graviton solutions carrying angular momenta in AdS_5 are constructed in [33], which takes advantage of the embedding of $AdS_5 \times S^5$ in $\mathbb{C}^{2+1} \times \mathbb{C}^3$ equipped with a flat metric with two negative signatures. Using the six complex coordinates of the latter space, the worldvolume of the giant graviton is given by the 6-dimensional holomorphic subspace defined by 3 complex equations which are homogeneous in a suitable sense. See [33] for the details. The latter space becomes 4-dimensional as one intersects this 6-dimensional space with AdS_5 and S^5. In particular, taking the 3 coordinates of $\mathbb{C}^3$ and $\mathbb{C}^{2+1}$ to be x, y, z and w_1, w_2, w_3 satisfying $|x|^2 + |y|^2 + |z|^2 = 1$ and $|w_3|^2 - |w_1|^2 - |w_2|^2 = 1$, the 3 complex equations for a spherical $\frac{1}{8}$-BPS dual giant gravitons can be written as

$$xw_3 = c_1\,, \quad yw_3 = c_2\,, \quad zw_3 = c_3\,, \tag{3.29}$$

where c_1, c_2, c_3 are constants. Note that one finds that $|w_3|^2 = c_1^2 + c_2^2 + c_3^2$ is fixed to be a constant, defining a 3-sphere in AdS_5. Fluctuations of the above configuration with angular momenta in AdS_5 are given by

$$w_3x\!=\!c_1\!+\!f_1(w_1w_3^{-1}, w_2w_3^{-1})\,,\ w_3y\!=\!c_2\!+\!f_2(w_1w_3^{-1}, w_2w_3^{-1})\,,\ w_3z\!=\!c_3\!+\!f_3(w_1w_3^{-1}, w_2w_3^{-1}) \tag{3.30}$$

where f_i are regarded as functions given by Taylor series of the arguments. As far as one considers small fluctuations from the spherical dual giants, $|f_i|^2 \ll |c_i|^2$, one finds that $|\omega_3|^2$ is approximately constant and (3.30) is nothing but

$$(x, y, z) \approx c_i e^{it} + e^{it} f_i(\hat{\omega}_1 e^{-it}, \hat{\omega}_2 e^{-it}) \tag{3.31}$$

where t is the time coordinate of the global AdS_5 defined by $w_3 = |w_3|e^{-it}$, and the hatted coordiantes satisfying $|\hat{\omega}_1|^2 + |\hat{\omega}_2|^2 = 1$ parametrize S^3 worldvolume of a dual giant.

Ignoring any possible correction as these fluctuations becomes large, one finds the following partition function for N' dual giant gravitons with fluctuations: firstly, the 1-particle (or a single dual giant) partition function is given by

$$\begin{aligned} \widetilde{Z}_1(\mu_i, \theta_a) &= \prod_{j_1, j_2=0}^{\infty} \frac{1}{(1-\mu_1\theta_1^{j_1}\theta_2^{j_2})(1-\mu_2\theta_1^{j_1}\theta_2^{j_2})(1-\mu_3\theta_1^{j_1}\theta_2^{j_2})} \\ &= \sum_{\{n^i_{j_1,j_2}\}=0}^{\infty} \prod_{i=1}^{3} \left(\mu_i^{\sum_{j_1,j_2=0}^{\infty} n^i_{j_1,j_2}} \theta_1^{\sum_{j_1,j_2=0}^{\infty} j_1 n^i_{j_1,j_2}} \theta_2^{\sum_{j_1,j_2=0}^{\infty} j_2 n^i_{j_1,j_2}} \right) . \end{aligned} \tag{3.32}$$

The partition function of N' or less identical dual giant gravitons is simply its multi-particling:

$$\begin{aligned} \widetilde{Z}(p, \mu_i, \theta_a) &= \sum_{N'=0}^{\infty} p^{N'} \widetilde{Z}_{N'}(\mu_i, \theta_a) \\ &= \prod_{\{n^i_{j_1,j_2}\}=0}^{\infty} \left(1 - p \prod_{i=1}^{3} \left(\mu_i^{\sum_{j_1,j_2=0}^{\infty} n^i_{j_1,j_2}} \theta_1^{\sum_{j_1,j_2=0}^{\infty} j_1 n^i_{j_1,j_2}} \theta_2^{\sum_{j_1,j_2=0}^{\infty} j_2 n^i_{j_1,j_2}} \right) \right)^{-1} , \end{aligned} \tag{3.33}$$

or

$$\widetilde{Z}_{N'}(\mu_i, \theta_a) = \sum_{\{p_k\},\ \sum k p_k = N'} \prod_{k=1}^{\infty} \frac{\widetilde{Z}_1\left({\mu_i}^k, {\theta_a}^k\right)^{p_k}}{p_k! k^{p_k}} . \tag{3.34}$$

The latter expression can be obtained by explicitly expanding the so-called Plethystic exponential containing the parameter p: see, for instance, eq. (2.9) in [24]. The expression (3.33) is what we used to evaluate the upper bound in the previous section (setting $e^{-\beta} = \mu_i = \theta_a$).

Now we consider the simple subsector studied in section 3.1. One can think of the states in this sector as $\frac{1}{8}$ BPS excitations of $\frac{1}{2}$ BPS states which carry two angular momenta J_1, J_2 in AdS_5, in addition to the charge H_3 conjugate to $\mu_3 \equiv \mu$. As mentioned in section 3.1, the same partition function has been obtained by quantizing the $\frac{1}{2}$ BPS giant gravitons with excitations[4] carrying nonzero angular momenta J_1, J_2 [21]. We now consider the subtleties involved in quantizing the $\frac{1}{2}$ BPS 'dual' giant gravitons with excitations carrying nonzero J_1, J_2.

The probe dual giant graviton description is reliable only for large N, or more precisely for $N' \ll N$ where N' is the number of dual giant gravitons. Following [34], we explore the

[4]Giant gravitons with such excitations are in fact $\frac{1}{8}$ BPS as discussed in section 3.1.

dual giant graviton interpretation of BPS states for the cases in which this condition is not obeyed, and consider the case with $N' \sim N$ or with small values of $N = 1, 2, 3 \ldots$, assuming that supersymmetry will help. Indeed, going beyond these limits, a prescription is found that the BPS states in $\mathrm{U}(N)$ gauge theory should come from N or less multiple dual giant gravitons [34, 21]. This is a prescription that we shall also assume in the sector we study. With this assumption, N' in the naive partition function of the previous paragraph is identified as N for the $\mathrm{U}(N)$ gauge theory. In the simplest case, $N = 1$, the naive partition function of a single dual giant graviton (3.32), with nonzero angular momenta, is simply

$$\widetilde{Z}_1(\mu, \theta_1, \theta_2) = \prod_{j_1, j_2 = 0}^{\infty} \frac{1}{1 - \mu \theta_1^{j_1} \theta_2^{j_2}} \,, \tag{3.35}$$

which is exactly the same as that of a $\mathrm{U}(1)$ (non-interacting) Yang-Mills theory.

For $N = 2$, we denote the exact, gauge theory partition function as $Z_2(\mu, \theta_a)$ and find that $\widetilde{Z}_2(\mu, \theta_a)$ and $Z_2(\mu, \theta_a)$ start to disagree. For simplicity, let us consider the states with $J_2 = 0$, or equivalently, states in the gauge theory with one kind of derivative only. A series expansion shows that

$$Z_2(\mu, \theta) = \prod_{j=0}^{\infty} \frac{1}{(1 - \mu \theta^j)(1 - \mu^2 \theta^j)} \tag{3.36}$$

$$\widetilde{Z}_2 - Z_2 = \prod_{j=0}^{\infty} \frac{1}{1 - \mu \theta^j} \Big(\theta^2 \mu^2 + \theta^3 (\mu^2 + \mu^4) + \theta^4 (2\mu^2 + 2\mu^4 + \mu^6) + \cdots \Big) \,. \tag{3.37}$$

We have left the overall $Z_1(\mu, \theta)$ factor unexpanded, which is the contribution from the decoupled overall $\mathrm{U}(1)$ mode. Therefore the series inside the parenthesis of (3.37) may be regarded as a result in the SU(2) gauge theory. A curious point we would like to emphasize is that, the coefficient of θ^j in the series is a finite polynomial which does not receive contributions from μ^{2j} or higher order terms. Therefore, considering the partition function of the SU(2) theory (which is also meaningful in the $\mathrm{U}(2)$ theory since the overall $\mathrm{U}(1)$ degrees never interact with others), the upper bound $\tilde{d}(H_3, J_1)$ (coefficient of $t^{H_3} \theta^{J_1}$) is equal to the true degeneracy $d(H_3, J_1)$ if $H_3 \geq 2 J_1$ is satisfied.

We would like to interpret the above (dis)agreement from the viewpoint of dual giant gravitons. Firstly, each of the two $\frac{1}{2}$ BPS dual giant gravitons can be described by a complex variable, say z introduced above, moving in a 2-dimensional harmonic potential [21]. The BPS fluctuations on these dual giant gravitons with nonzero angular momentum J_1 explained above can be regarded as a spherical harmonics expansion of $z = \sum_{j=0}^{\infty} z_j Y_{j0}$ in the basis $Y_{j0} \sim (\hat{\omega}_1)^j$. Inserting this expansion in the world-volume action, the harmonic potential for the modes z_j becomes

$$\frac{1}{2} \sum_{j=0}^{\infty} \left(|z_j|^2 + j(j+2) |z_j|^2 \right) = \frac{1}{2} \sum_{j=0}^{\infty} (j+1)^2 |z_j|^2 \,. \tag{3.38}$$

For two dual giant gravitons and their fluctuations, one has two towers of modes z_j^1 and z_j^2. One can separate the dynamics of the modes $\frac{z_j^1 + z_j^2}{2}$, which gives the overall $\mathrm{U}(1)$ partition function factor $Z_1(\mu,\theta)$ in $\widetilde{Z}_2$, and the modes of relative separation $z_j \equiv z_j^1 - z_j^2$. We concentrate on the latter part. After naive quantization, the charges coming from the relative motion degrees of freedom are,

$$H_3 = \sum_j N_j\,, \quad J_1 = \sum_j jN_j \qquad \left(N_j \equiv \frac{j+1}{2}|z_j|^2\right)\,. \tag{3.39}$$

From the above expressions, we now argue that $H_3 \sim J_1$ is the region in charge space where the two world-volumes of the dual giant gravitons become likely to intersect due to the fluctuations. To see this, we note that, with fixed angular momentum J_1, the fluctuation of the relative separation (governed by z_j with nonzero j) becomes biggest if we assign more occupations to the modes with lowest nonzero value of j, namely $j = 1$. The case with biggest fluctuation is thus obtained by assigning $N_1 = J_1$, $N_0 = H_3 - N_1 = H_3 - J_1$ and all other N_j's zero. The separation $|z_0|$ of two dual giants averaged over S^3, and its fluctuation $|z_1|$ are given by

$$|z_0| = \sqrt{2N_0} = \sqrt{2(H_3 - J_1)}\,, \quad |z_1| = \sqrt{N_1} = \sqrt{J_1}\,. \tag{3.40}$$

Demanding $|z_0| \gtrsim |z_1|$ for the two worldvolumes not to intersect, one finds the condition $H_3 \gtrsim J_1$ which is qualitatively similar to the condition $H_3 \geq 2J_1$ for the upper bound to be exact. Therefore, we interpret that our naive quantization of multiple dual giant gravitons becomes invalid as a dual giant graviton becomes close to, or intersects (in a singular way) with another.

The SU(2) example above is rather special since there are two charges H_3 and J_1 which can be used to control both the average separation r_0 of two branes and the fluctuation of its relative separation. For $N \geq 3$, it becomes impossible to control all the relative separations with these two conserved charges only. Indeed, as we will see shortly, we could not identify any region in the charge space where the upper bound becomes exact (except for some exceptional and rather occasional values of charges). However, we would still like to argue that our upper bound becomes approximately valid in certain regimes of charges in which the dual giant gravitons are less likely to intersect.

For the cases with $N \geq 3$, we still consider the regime $H_3 \gg 1$. In the exact partition function $Z(\mu,\theta)$ and the upper bound partition function $\widetilde{Z}(\mu,\theta)$, the asymptotic growth of degeneracy as a function of the charge H_3 is captured by studying the degree of the pole of the partition function as $\mu \to 1$. Namely, the Taylor series expansion

$$\frac{1}{(1-\mu)^\alpha} = \frac{1}{(\alpha-1)!}\left(\frac{d}{d\mu}\right)^{\alpha-1}\sum_{n=0}^{\infty}\mu^n = \frac{1}{(\alpha-1)!}\sum_{n=0}^{\infty}\frac{(n+\alpha-1)!}{n!}\mu^n \tag{3.41}$$

grows like $\sim n^{\alpha-1}\mu^n$ for terms with large n, while the other μ-dependent functions without poles, in our case, turn out to contribute to the coefficient with alternating signs. Therefore, with poles of degree α in the partition function, the asymptotic growth of the degeneracy with large charge H_3 should grow like $(H_3)^{\alpha-1}$.[5]

[5] The same conclusion can be obtained by the saddle point method, showing that the saddle point is at $H_3 \approx \frac{\alpha}{1-\mu}$ when $H_3 \gg 1$.

From a simple computation, one can show that

$$Z_N(\mu,\theta) = \frac{Z_1(\mu,\theta)}{(1-\mu)^{N-1}}\left(\frac{1}{N!}\prod_{j=1}^{\infty}\frac{1}{(1-\theta^j)^{N-1}}\right) \tag{3.42}$$
$$\times\left(1+(1-\mu)\left(\frac{N(N-1)}{4}-\frac{(N+2)(N-1)}{2}\sum_{j=1}^{\infty}\frac{\theta^j}{1-\theta^j}\right)+O(1-\mu)^2\right)$$

$$\widetilde{Z}_N(\mu,\theta) = \frac{Z_1(\mu,\theta)}{(1-\mu)^{N-1}}\left(\frac{1}{N!}\prod_{j=1}^{\infty}\frac{1}{(1-\theta^j)^{N-1}}\right) \tag{3.43}$$
$$\times\left(1+(1-\mu)\left(\frac{N(N-1)}{4}\prod_{j=1}^{\infty}\frac{1-\theta^j}{1+\theta^j}-(N-1)\sum_{j=1}^{\infty}\frac{\theta^j}{1-\theta^j}\right)+O(1-\mu)^2\right)$$

where $Z_1(\mu,\theta)=\prod_{j=0}^{\infty}\frac{1}{1-\mu\theta^j}$ is again the contribution from the overall U(1) part which we shall ignore in the arguments below. From these expansions we see that, for large H_3 (or equivalently near $\mu\approx 1^-$), the two degeneracies (or the partition functions) are asymptotically the same. Both degeneracies grow as $\sim (H_3)^{N-2}f(J)$, while their difference grows more slowly as $\sim (H_3)^{N-3}g(J)$ since

$$\widetilde{Z}_N - Z_N = \frac{Z_1(\mu,\theta)}{2(N-2)!(1-\mu)^{N-2}}\prod_{j=1}^{\infty}\frac{1}{(1-\theta^j)^{N-1}}\left(\sum_{j=1}^{\infty}\frac{\theta^j}{1-\theta^j}\right. \tag{3.44}$$
$$\left.-\frac{1}{2}+\frac{1}{2}\prod_{j=1}^{\infty}\frac{1-\theta^j}{1+\theta^j}\right)+\cdots\ (>0)\ .$$

The J-dependent functions $f(J)$ and $g(J)$ are determined from the θ dependent coefficients in the above expansions.

We argue that the above scaling behaviors can be qualitatively reproduced from the dynamics of multiple dual giant gravitons. Again we keep J_1 to be much smaller than $H_3\gg 1$. The H_3 dependent part of the degeneracy $d(H_3,J_1)$ will then be simply determined by the dynamics of $\frac{1}{2}$-BPS dual giant gravitons, while the J-dependent factor can be determined by investigating the fluctuations on the $\frac{1}{2}$-BPS dual giants. The phase space of N $\frac{1}{2}$-BPS giant gravitons is an N-th symmetric product of $\mathbb{C}$. Semiclassically, the number of states with fixed charge $2H_3=|\vec{r}_1|^2+|\vec{r}_2|^2+\cdots+|\vec{r}_N|^2$ is obtained by computing the volume of the region in the phase space with fixed H_3. Taking into account the SU(N) condition, which keeps the contribution from the relative seperation $\vec{r}_i-\vec{r}_j$ only, the volume is proportional to

$$d(H_3,J)\sim\int d^2\vec{r}_1\cdots d^2\vec{r}_{N-1}\delta\left(2H_3-\sum_{i=1}^{N}|\vec{r}_i|^2\right)\sim (H_3)^{N-2}\ . \tag{3.45}$$

On the other hand, the naive dual giant graviton counting should go significantly wrong as the two dual giant gravitons become close, namely, $|\vec{r}_i-\vec{r}_j|^2\lesssim\epsilon(J)$ for any pair $i,j=$

$1, \ldots, N$ and certain $\epsilon(J)$ whose value is roughly given by the fluctuations of S^3. The volume of the region with close enough dual giant gravitons should go as

$$\tilde{d}(H_3, J_1) - d(H_3, J_1) \sim \int d^2\vec{r}_1 \cdots d^2\vec{r}_{N-1}\delta\left(2H_3 - \sum_{i=1}^{N} |\vec{r}_i|^2\right)\theta\left(\epsilon(J_1) + \right. \qquad (3.46)$$
$$\left. -|\vec{r}_i - \vec{r}_j|^2\right) \sim \epsilon(J_1)(H_3)^{N-3}.$$

where $\theta(x)$ is the step function $\theta(x) = 1$ for $x > 0$ and zero otherwise. Thus, the qualitative feature of the error in the naive dual giant counting, described in the previous paragraph, is the same as what we expect from the intersection of dual giants.

From the above examples — namely the exactness of U(1) result, exactness of SU(2) result in certain regime, and also the semiclassical error estimate for $N \geq 3$ — it seems that the overcounting one gets by naively quantizing dual giant gravitons has to do with ignoring their intersections as fluctuations become large. Note also that, in the giant graviton counting of [21] which gave us the correct answer, a giant graviton never intersects with another. It would be interesting to see if the dual giant graviton like counting is available which should modify our naive upper bound partition function. For instance, it may be possible that restricting the world-volume of multiple dual giants to intersect smoothly, in a suitable sense, could give the correct answer $Z_N(\mu, \theta)$. Otherwise, it could turn out that non-Abelian physics would be important as the dual giant gravitons intersect.

Before concluding this subsection, let us mention a few results for the SU(2) case. In this case, even for the operators including all three scalasrs, our upper bound is saturated for large enough internal charges H_1, H_2, H_3. Namely, we find that $\max(H_1, H_2, H_3) \geq 2J_1$ is a sufficient condition for the upper bound $\tilde{d}(H_i, J_1)$ to be exact. This claim can be shown analytically by induction: assuming that the claim is proved for operators in which no more than k derivatives act on same eigenvalues, it is easy to check that the same claim can be shown for operators in which no more than $k+1$ derivativese act on same eigenvalues. The claim for $k = 1$ is easily proved from the condition $\max(H_1, H_2, H_3) \geq 2J_1$: for instance if H_1 is the largest, one can rewrite ∂x, ∂y, ∂z into $\partial(x^2)$, $\partial(xy)$, $\partial(xz)$, respectively, using abundant x's in the operator. By investigating the case in which an exact partition function is available, we find that the condition $\max(H_1, H_2, H_3) \geq 2J_1$ is indeed a sufficient but may not be a necessary condition for the upper bound to be exact. For example, in the SU(2) two-scalar case in which we have an exact partition function, one finds:

$$\begin{aligned}
\widetilde{Z}_2(\mu_i, \theta) - Z_2(\mu_i, \theta) = \ & \theta(\mu_1\mu_2) + \theta^2(\mu_1^3\mu_2 + \mu_1^2\mu_2^2 + \mu_1\mu_2^3 + \mu_1^2 + 2\mu_1\mu_2 + \mu_2^2) \qquad (3.47)\\
& +\theta^3(\mu_1^5\mu_2 + \mu_1^4\mu_2^2 + \mu_1^3\mu_2^3 + \mu_1^2\mu_2^4 + \mu_1\mu_2^5\\
& \qquad +\mu_1^4 + 3\mu_1^3\mu_2 + 3\mu_1^2\mu_2^2 + 3\mu_1\mu_2^3 + \mu_2^4 + \mu_1^2 + 3\mu_1\mu_2 + \mu_2^2)\\
& +\theta^4(\mu_1^7\mu_2 + \mu_1^6\mu_2^2 + \mu_1^5\mu_2^3 + \mu_1^4\mu_2^4 + \mu_1^3\mu_2^5 + \mu_1^2\mu_2^6 + \mu_1\mu_2^7 +\\
& \qquad +\mu_1^6 + 3\mu_1^5\mu_2 + 4\mu_1^4\mu_2^2 + 4\mu_1^3\mu_2^3 + 4\mu_1^2\mu_2^4 + 3\mu_1\mu_2^5 + \mu_2^6\\
& \qquad +2\mu_1^4 + 6\mu_1^3\mu_2 + 7\mu_1^2\mu_2^2 + 6\mu_1\mu_2^3 + 2\mu_2^4 + 2\mu_1^2 + 4\mu_1\mu_2 + 2\mu_2^2)\\
& +O(\theta^5)\ .
\end{aligned}$$

For instance, terms like $\theta^3\mu_1^4\mu_2^4$ or $\theta^4\mu_1^7\mu_2^3$ which do not satisfy the bound $\max(H_1, H_2) \geq 2J_1$ and are not forbidden by gauge-invariance, happen to be zero. It would be interesting to find a stronger bound which admits a better geometric interpretation in terms of dual giant gravitons.

4. $\frac{1}{16}$-BPS classical configurations of $\mathcal{N}=4$ Yang-Mills

In the rest of this paper we switch gears rather abruptly. Instead of pursuing more sophisticated countings of Q cohomology in larger subsectors, we attempt to find a physical interpretation of the mathematically motivated generating functions, $\bar{\phi}^m(z)$ that we introduced in section 2 and have used intensively since then. For this purpose we take a step backwards; we turn off all fermions and study classical $\mathcal{N}=4$ Yang Mills theory. We will derive a set of Bogomolnyi type equations for the bosonic $\frac{1}{16}$-BPS configurations in $\mathcal{N}=4$ Yang-Mills theory. We will analyze these equations in perturbation theory, propose a quantization of the bosonic $\frac{1}{16}$-BPS configurations and find that we recover a characterization equivalent to that of the 1/16 BPS cohomology found in section 2.3. In this section, we will derive the BPS equations in a convenient gauge. Similar analysis and quantization have been studied in a simpler $\frac{1}{4}$- and $\frac{1}{8}$-BPS sector of M5 and M2 brane CFT's, respectively [35].

4.1 $\frac{1}{16}$-BPS equations of $\mathcal{N}=4$ Yang-Mills on $S^3\times\mathbb{R}$

For convenience, we denote the 3 chiral scalars as ϕ^i, where $i=1,2,3$ and we will not use any lowered SU(3) indices. Rather, an SU(3) index on a conjugate field, $\bar{\phi}^i$, will be understood as anti-fundamental. We begin by considering the theory on $S^3\times\mathbb{R}$.

The bosonic part of the action of $\mathcal{N}=4$ Yang-Mills theory on $S^3\times\mathbb{R}$ is

$$S=\int dtd^3\Omega\,\mathrm{tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}D_\mu\phi^i D^\mu\bar{\phi}^i-\frac{1}{2}|\phi^i|^2+\frac{1}{4}\left[\phi^i,\phi^j\right]\left[\bar{\phi}^i,\bar{\phi}^j\right]-\frac{1}{8}\left[\phi^i,\bar{\phi}^i\right]^2\right), \tag{4.1}$$

and the energy density is

$$\mathcal{E} = \mathrm{tr}\left[\frac{1}{2}(F_{03})^2+\frac{1}{2}(F_{12})^2+\frac{1}{2}\Big((F_{a0})^2+(F_{a3})^2\Big)\right. \tag{4.2}$$

$$+\frac{1}{2}\Big(|D_0\phi^i|^2+|D_3\phi^i|^2\Big)+\frac{1}{2}\Big(|D_1\phi^i|^2+|D_2\phi^i|^2\Big)+\frac{1}{2}|\phi^i|^2 \tag{4.3}$$

$$\left.-\frac{1}{4}[\phi^i,\phi^j][\bar{\phi}^i,\bar{\phi}^j]+\frac{1}{8}[\phi^i,\bar{\phi}^i]^2\right]. \tag{4.4}$$

Here $i=1,2,3$, and the $a=1,2$ and 3 subscripts label a local orthonomal frame on S^3, where the dreibein may be chosen to be proportional to the left-invariant 1-forms

$$\begin{aligned} e^1 &= \sin\psi d\theta-\cos\psi\sin\theta d\phi\\ e^2 &= \cos\psi d\theta+\sin\psi\sin\theta d\phi\\ e^3 &= d\psi+\cos\theta d\phi\ . \end{aligned} \tag{4.5}$$

We define $\phi^i \equiv X^i + iY^i$ with X^i, Y^i hermitian[6] and by a suitable choice of the time coordinate, we take the scalar mass to be 1.

It is shown in appendix C.1 that the energy density can be written as:

$$
\begin{aligned}
\mathcal{E} = \operatorname{tr}\Bigg[& \frac{1}{2}\left(F_{12} + \frac{1}{2}[\phi^i, \bar{\phi}^i]\right)^2 + \frac{1}{2}\Big(\, F_{a0} - F_{a3}\Big)^2 + \frac{1}{2}\Big|\, D_0\phi^i - D_3\phi^i + i\phi^i\Big|^2 \\
& + \frac{1}{2}\Big|\,(D_1 + iD_2)\phi^i\Big|^2 + \frac{1}{2}(F_{03})^2 + \frac{1}{4}\Big|\ [\phi^i, \phi^j]\Big|^2 \\
& + \left(F_{a0}F_{a3} + \frac{1}{2}\left(D_0\phi^i D_3\bar{\phi}^i + D_0\bar{\phi}^i D_3\phi^i\right)\right) + \frac{i}{2}\left(\bar{\phi}^i D_0\phi^i - \phi^i D_0\bar{\phi}^i\right)\Bigg] \ .
\end{aligned}
\tag{4.6}
$$

The last line of equation (4.6) is a sum of the conserved charges of this theory:

$$
J \equiv \frac{1}{2}\operatorname{tr}\left[F_{a0}F_{a3} + \frac{1}{2}\left(D_0\phi^i D_3\bar{\phi}^i + D_0\bar{\phi}^i D_3\phi^i\right)\right] \tag{4.7}
$$

is a component of the SO(4) the angular momentum conjugate to ψ, and

$$
Q_i \equiv \frac{i}{2}\operatorname{tr}\left(\bar{\phi}^i D_0\phi^i - \phi^i D_0\bar{\phi}^i\right) \qquad \text{(no sum over } i\text{)} \tag{4.8}
$$

are the three $\mathrm{U}(1)^3 \subset \mathrm{SO}(6)$ R-charges. For given values of these charges, the energy is minimized when the complete-squared terms on the first and second lines of (4.6) vanish. The Bogomolnyi equations obtained in this way are

$$
F_{12} + \frac{1}{2}[\phi^i, \bar{\phi}^i] = 0\,, \quad F_{0a} = F_{3a}\,, \quad F_{03} = 0 \tag{4.9}
$$

and

$$
[\phi^i, \phi^j] = 0\,, \quad D_0\phi^i - D_3\phi^i + i\phi^i = 0\,, \quad (D_1 + iD_2)\phi^i = 0\ . \tag{4.10}
$$

The energy of a configuration satisfying these equations is

$$
\mathcal{E} = 2J + \sum_{i=1}^{3} Q_i\ . \tag{4.11}
$$

This is a classical version of equation (2.4) so that configurations satisfying the Bogomolnyi equations above preserve the supersymmetry generated by a single supercharge and its Hermitian conjugate.

Apart from the above set of BPS equations, we should also impose the Gauss law constraint to ensure the configuration solves all the equations of motion. The Gauss law constraint

$$
D^\mu F_{\mu 0} + \frac{i}{2}\Big([\phi^i, D_0\bar{\phi}^i] + [\bar{\phi}^i, D_0\phi^i]\,\Big) = 0 \tag{4.12}
$$

[6] We note that the scalar potential of $\mathcal{N} = 4$ Yang-Mills can be decomposed into the F- and D- term potentials as

$$
-\frac{1}{4}\operatorname{tr}\Big(\,[X^i, X^i]^2 + 2[X^i, Y^i]^2 + [Y^i, Y^i]^2\Big) = -\frac{1}{4}\operatorname{tr}[\phi^i, \phi^j][\bar{\phi}^i, \bar{\phi}^j] + \frac{1}{8}\operatorname{tr}[\phi^i, \bar{\phi}^i]^2\,,
$$

which we used in the expressions (4.1) and (4.2) for the Lagrangian and energy density.

can be rewritten using some of the BPS equations as

$$D^a F_{a3} + \frac{i}{2}\Big([D_3\phi^i, \bar{\phi}^i] - [\phi^i, D_3\bar{\phi}^i]\Big) + [\phi^i, \bar{\phi}^i] = 0 \,. \tag{4.13}$$

Here the covariant derivative D_a contains spin connection as well as the Yang-Mills connection.

4.2 Reformulation: configurations in $\mathbb{R}^4$

The $\mathcal{N} = 4$ Yang-Mills theory $S^3 \times \mathbb{R}$ can be mapped to the theory defined on $\mathbb{R}^4$ by a conformal tansformation which places time along the radial direction of $\mathbb{R}^4$. It will be convenient to consider the BPS equations that we derived in the this framework. In this subsection, after reviewing the map between classical configurations in two theories, we rewrite the BPS equations for configurations in $\mathbb{R}^4$.

We first introduce $\tau = it$, where t is the $S^3 \times \mathbb{R}$ time coordinate. This changes the positive frequency modes e^{-it} into $e^{-\tau}$. Then we identify the radial direction, r, of $\mathbb{R}^4$ with τ through $r \equiv e^{\tau}$. The Lagrangian on $S^3 \times \mathbb{R}$ should change to that on $\mathbb{R}^4$ ($iS = -S_E$) where the fields are related by

$$[\phi^i]_{S^3} = r[\phi^i]_{R^4}\,, \quad [A_m]_{S^3} = [A_m]_{R^4} \quad (m = 1, 2, 3)\,, \quad [A_0]_{S^3} = i[A_\tau]_{S^3} = ir[A_r]_{R^4} \,. \tag{4.14}$$

Note that we are *not* considering the analytically continued theory: Euclidean notations are used since it simplifies the analysis, but we are still considering the *real-time* physics of this theory by regarding τ and A_τ as imaginary.

We introduce complex coordinates (Z^1, Z^2) which are related to the spherical coordinates, (r, θ, ψ, ϕ), of equation (4.5) as

$$\begin{aligned} Z^1 &= r\cos\zeta\ e^{i\frac{\psi+\phi}{2}} \\ Z^2 &= r\sin\zeta\ e^{i\frac{\psi-\phi}{2}}\,, \end{aligned} \tag{4.15}$$

where $\zeta = \theta/2$. The relation between derivatives in these two coordinate systems can be found in appendix D. Written in the new coordinates which cover $\mathbb{R}^4$, the BPS equations become:

$$F_{\bar{1}\bar{2}} = 0\,, \qquad F_{I\bar{J}}\bar{Z}^J = 0\,, \qquad F_{1\bar{1}} + F_{2\bar{2}} + \frac{i}{4}[\phi^i, \bar{\phi}^i] = 0 \tag{4.16}$$

$$D_{\bar{I}}\phi^i = 0\,, \qquad [\phi^i, \phi^j] = 0 \tag{4.17}$$

where $I = 1, 2$. We note the distinctive fact that complex conjugation on $S^3 \times \mathbb{R}$ becomes complex conjugation plus radial inversion on $\mathbb{R}^4$ because of the relation $r \equiv e^{\tau}$ and of the fact that τ is imaginary. Therefore, when complex conjugating in $\mathbb{R}^4$, we should simultaneously perform a coordinate inversion $x'^{\mu} = \frac{x^\mu}{r^2}$. With this in mind, we define a new conjugation operation as $[f(Z^I)]^\star \equiv f^*(\bar{Z}_I/r^2)$. The gauge field transforms under inversion like a derivative, $\partial_\mu = \frac{1}{r^2}\partial'_\mu - \frac{2x_\mu x^\nu}{r^4}\partial'_\nu$, so the reality constraint, $A_\mu = A^*_\mu$, of the gauge field in $S^3 \times \mathbb{R}$ is modified to:

$$\begin{aligned} A_\mu &= \frac{1}{r^2}A^\star_\mu - \frac{2x_\mu x^\nu}{r^4}A^\star_\nu\,, \\ F_{\mu\nu} &= \frac{1}{r^4}F^\star_{\mu\nu} - \frac{2}{r^6}\left(x_\mu x^\rho F^\star_{\rho\nu} + x_\nu x^\rho F^\star_{\mu\rho}\right)\,, \end{aligned} \tag{4.18}$$

where $F^\star_{\mu\nu} = \partial'_\mu A^\star_\nu - \partial'_\nu A^\star_\mu$. Applying the BPS equations, we can write the complex conjugation of the scalar and field strength in $\mathbb{R}^4$ as:

$$\bar{\phi}^i = \frac{1}{r^2}(\phi^i)^\star \tag{4.19}$$

$$F_{I\bar{J}} = \frac{\bar{Z}^I\left(\epsilon_{\bar{J}\bar{K}}\bar{Z}^K\right)}{r^6}(F_{12})^\star - \frac{i}{4}\frac{\left(\epsilon_{IK}Z^K\right)\left(\epsilon_{\bar{J}\bar{L}}\bar{Z}^L\right)}{r^2}[\phi^i,\bar{\phi}^i]\ . \tag{4.20}$$

The equation (4.16) relate most of the components of the field strength so that the only independent components of the field strength in a BPS configuration may be taken to be F_{12} and $F_{1\bar{1}}$. These components are further related by the reality constraint (4.20).

The details involved in obtaining the Gauss' law constraint in $\mathbb{R}^4$ are relegated to appendix C.2 and we list only the final constraint here. Defining curly derivatives as

$$\mathcal{D} \equiv \frac{1}{r^2}\left(\bar{Z}^2 D_1 - \bar{Z}^1 D_2\right)\ ,\quad \bar{\mathcal{D}} \equiv r^2\left(Z^2 D_{\bar{1}} - Z^1 D_{\bar{2}}\right)\ , \tag{4.21}$$

the Gauss law constraint is:

$$\bar{\mathcal{D}}F_{12} + \frac{i}{4}\left[\phi^i, Z\cdot D\phi^{i^\star}\right] - \frac{i}{4}\left[\phi^i,\phi^{i^\star}\right] = 0\ . \tag{4.22}$$

4.3 Axial gauge

In this section, we will make a convenient choice of gauge which solves some of the BPS relations and reduces the number of constraints to be considered. The boundary conditions appropriate for fields in the radial quantization will also play role in constraining the BPS solutions.

We make the following choice of gauge:

$$\bar{Z}^I A_{\bar{I}} = 0\ . \tag{4.23}$$

With this choice of gauge, we find several simplifications. First, the condition $F_{I\bar{J}}\bar{Z}^J = 0$ becomes

$$\bar{Z}\cdot\bar{\partial}A_I = 0\,, \tag{4.24}$$

which says that A_I should be degree 0 in $\bar{Z}$. We will restrict our interest to the configurations A_I admitting radial quantization, namely, those having poles only at 0 or ∞ corresponding to $t = \pm\infty$. Then, for A_I to be of anti-holomorphic degree 0, A_I must be a power series in Z^I and $\frac{\bar{Z}^I}{r^2}$. Furthermore, one finds that the condition $F_{\bar{1}\bar{2}} = 0$ becomes linear in $A_{\bar{I}}$ because our gauge condition $\bar{Z}^I A_{\bar{I}} = 0$ implies that $A_{\bar{1}}$ and $A_{\bar{2}}$ are proportional to each other as matrices so that $[A_{\bar{1}}, A_{\bar{2}}] = 0$. The resulting linear condition implies

$$\partial_{\bar{1}}A_{\bar{2}} - \partial_{\bar{2}}A_{\bar{1}} = 0 \ \rightarrow\ A_{\bar{I}} = \partial_{\bar{I}}v \tag{4.25}$$

for some matrix v and the gauge condition now becomes

$$\bar{Z}\cdot\bar{\partial}v = 0\ . \tag{4.26}$$

This means that v is degree 0 in $\bar{Z}$, so that $A_{\bar{I}}$ is degree -1 in $\bar{Z}$. Again allowing $A_{\bar{I}}$ to have poles only at 0 and ∞, we find that $A_{\bar{I}}$ is $\frac{1}{r^2}$ times a series expansion of Z^I and $\frac{\bar{Z}^I}{r^2}$.[7]

Putting together the above observations and the gauge condition $\bar{Z}^I A_{\bar{I}} = 0$, the potential $A_{\bar{I}}$ takes the form

$$A_{\bar{I}} = \frac{\epsilon_{\bar{I}\bar{J}} \bar{Z}^J}{r^4} f^\star \left(\frac{\bar{Z}}{r^2}, Z \right) , \tag{4.27}$$

where $f^\star$ is an arbitrary function taking the form of series expansion of the arguments. The most general form of A_I, compatible with the degree constraint and also with (4.27) through complex conjugation, is given by

$$A_I = i \frac{\bar{Z}^I}{r^2} g \left(Z, \frac{\bar{Z}}{r^2} \right) + \epsilon_{IJ} Z^J f \left(Z, \frac{\bar{Z}}{r^2} \right) , \tag{4.28}$$

where g is an arbitrary Hermitian matrix function with respect to the $\star$ operation.

To summarize, we have chosen a gauge, solved $F_{\bar{I}\bar{J}} \bar{Z}^J = 0$, and expressed all components of the gauge field in terms of a function f and a Hermitian function g. The remaining equations to be solved are

$$F_{1\bar{1}} + F_{2\bar{2}} + \frac{i}{4} \left[\phi^i, \bar{\phi}^i \right] = 0 \, , \quad D_{\bar{I}} \phi^i = 0 \, , \quad \left[\phi^i, \phi^j \right] = 0 \tag{4.29}$$

and the Gauss' Law

$$\bar{\mathcal{D}} F_{12} + \frac{i}{4} \left[\phi^i, Z \cdot D \phi^{i\star} \right] - \frac{i}{4} \left[\phi^i, \phi^{i\star} \right] = 0 \, . \tag{4.30}$$

These two equations are nonlinear differential equations of the functions f, g and ϕ^i, which we expect to be difficult to solve in general. Nevertheless, a class of exact solutions to these equations can be obtained by imposing additional symmetry requirements. These solutions are described in appendix E.

In the next section, we try to analyze the equations (4.29) and (4.30) approximately in the weakly-coupled regime, in which the functions are expanded into power series of g_{YM}.

5. Classical $\frac{1}{16}$-BPS configurations in weakly interacting theory

In this section we analyze the differential conditions for the supersymmetric configurations perturbatively in the weakly-coupled theory. For the sake of convenience, we first consider the sector with only gauge fields in section 5.1, and then generalize to configuration involving nonzero scalars in section 5.2.

5.1 Perturbative expansion in g_{YM} with scalars set to zero

We will now use our gauge choice and write out the BPS equations (4.29) and Gauss law (4.30) in terms of the 2 functions $g(Z, \bar{Z}/r^2)$ and $f(Z, \bar{Z}/r^2)$ which appear in the gauge

[7] This does not necessarily mean that the function v should also be a series expansion of Z^I and $\frac{\bar{Z}^I}{r^2}$.

potential in equations (4.27) and (4.28). Then we will expand f and g in terms of the coupling constant $g_{\rm YM}$. We define differential operators as

$$\delta \equiv \frac{1}{r^2}\epsilon^{IJ}\bar{Z}^J\partial_I\,, \qquad\qquad \bar{\delta} \equiv r^2\epsilon^{\bar{I}\bar{J}}Z^J\partial_{\bar{I}}\,, \tag{5.1}$$

$$\delta\bar{\delta} = r^2\partial_I\partial_{\bar{I}} - (\bar{Z}\cdot\bar{\partial}) - (Z\cdot\partial)(\bar{Z}\cdot\bar{\partial})\,, \tag{5.2}$$

$$\bar{\delta}\delta = r^2\partial_I\partial_{\bar{I}} - (Z\cdot\partial) - (Z\cdot\partial)(\bar{Z}\cdot\bar{\partial})\,. \tag{5.3}$$

For simplicity, we use this section to record the BPS equations and Gauss law for configurations where the scalars are turned off and illustrate the perturbative expansion in that context. Writing the BPS and Gauss law equations (4.29) and (4.30) in terms of the functions f, g gives:

$$\begin{aligned} 0 =& g + [f, f^\star] + i\left(\delta f^\star - \bar{\delta} f\right) \\ 0 =& -2\bar{\delta} f - \bar{\delta}(Z\cdot\partial)f + i\bar{\delta}\delta g + \bar{\delta}[f, g] \\ & -2i[f, f^\star] + i[f^\star, Z\cdot\partial f] + [f^\star, \delta g] - i[f^\star, [f, g]]\,. \end{aligned} \tag{5.4}$$

One can explicitly check, by using the first equation, that the second equation is purely imaginary. We now treat the Yang-Mills coupling constant as small expansion parameter. To restore $g_{\rm YM}$ in the above equations which are all omitted, it suffices to put one $g_{\rm YM}$ in front of each commutator. The functions f and g are expanded as

$$f = f_0 + g_{\rm YM}f_1 + (g_{\rm YM})^2 f_2\cdots\,, \quad g = g_0 + g_{\rm YM}g_1 + (g_{\rm YM})^2 g_2\cdots\,. \tag{5.5}$$

The gauge transformation acts as

$$f \to U^\star f U + \frac{i}{g_{\rm YM}}U^\star\delta U\,, \quad g \to U^\star g U + \frac{1}{g_{\rm YM}}U^\star(Z\cdot\partial)U\,, \tag{5.6}$$

where the unitary matrix U can be written as

$$U = e^{-ig_{\rm YM}u}\,, \quad u = u_0 + g_{\rm YM}u_1 + (g_{\rm YM})^2 u_2\cdots\,. \tag{5.7}$$

The solution of the free theory is:

$$f = f_0(Z) + \mathcal{O}(g_{\rm YM})\,, \quad g = 0 + \mathcal{O}(g_{\rm YM})\,, \tag{5.8}$$

where $f_0(Z)$ is an $N\times N$ matrix whose components are holomorphic functions of $Z^{1,2}$. The free BPS solution is therefore given by N^2 holomorphic functions.

The differential equations at the next order $\mathcal{O}(g_{\rm YM})$ are:

$$g_1 = [f_0^\star, f_0] + i\left(\bar{\delta} f_1 - \delta f_1^\star\right)\,, \tag{5.9}$$

$$F_{12}^{(1)} = -(2 + Z\cdot\partial)f_1 + i[f_0^\star, \delta f_0] + \delta\delta f_1^\star - \delta\bar{\delta} f_1\,. \tag{5.10}$$

Taking the first equation in (5.9) to solve for g_1, we write out the $\mathcal{O}(g_{\rm YM})$ part of the second equation as a condition on f_1 and $f_1^\star$:

$$\delta\bar{\delta}\bar{\delta} f_1 - \bar{\delta}\delta\delta f_1^\star = i\left[\bar{\delta} f_0^\star, \delta f_0\right] + 2i\left[f_0^\star, f_0\right]\,. \tag{5.11}$$

In the free theory, the right hand side of (5.11) vanishes because $g_{\rm YM} = 0$ and the zeroth order solution $f_0(Z)$ (with $g_0 = 0$) solves the BPS equations exactly. We shall explain below that not all solutions of the free BPS equations can be perturbed to BPS solutions of (5.11) at nonzero coupling. In fact, for general $f_0(Z)$, we will show that there can be obstructions to the existence of solutions, f_1, to (5.11). For such obstructions to be absent, $f_0(Z)$ will have to satisfy a set of constraints which we list below.

Arguments in appendix F.1 show that the integrability constraint on the right hand side of equation (5.11) is that it may contain all terms, $(Z^1)^a(Z^2)^b\left(\frac{\bar{Z}^1}{r^2}\right)^c\left(\frac{\bar{Z}^2}{r^2}\right)^d$ except those where $a = b = 0$ or $c = d = 0$. That is $\delta\bar{\delta}h$ (where $h = \bar{\delta}f_1$) should not contain any holomorphic (or their $\star$ conjugate) terms.

To process this constraint, we must collect all purely holomorphic terms which may appear in:

$$\left[\bar{\delta}f_0^\star, \delta f_0\right] + 2\left[f_0^\star, f_0\right] \ . \tag{5.12}$$

We do this by expanding the zeroth order solution $f_0(Z)$ as

$$f_0(Z) = \sum_{n_1,n_2=0}^{\infty} a_{n_1n_2} Y_{n_1n_2}(Z^1, Z^2) \, , \tag{5.13}$$

where the coefficients $a_{n_1n_2}$ are matrix-valued, and the functions

$$Y_{n_1n_2} \equiv \sqrt{\frac{(n_1+n_2+1)!}{2\pi^2(n_1)!(n_2)!}}\,(Z^1)^{n_1}(Z^2)^{n_2} \equiv C_{n_1n_2}(Z^1)^{n_1}(Z^2)^{n_2} \tag{5.14}$$

are normalized on unit S^3 (i.e., $r = 1$) as

$$\int_0^{\frac{\pi}{2}} d\zeta \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \ (Y_{m_1m_2})^\star Y_{n_1n_2}\Big|_{r=1} = \delta_{m_1n_1}\delta_{m_2n_2} \ . \tag{5.15}$$

The full holomorphic contribution to the right hand side of equation (5.11) is calculated in appendix F.2:

$$\sum_{n_1,n_2=0}^{\infty} Y_{n_1n_2}(Z^1, Z^2) \sum_{k_1,k_2=0}^{\infty} (k_1+k_2+2)\,\frac{C_{n_1n_2}C_{k_1k_2}}{C_{n_1+k_1,n_2+k_2}}\left[a^*_{k_1k_2},\, a_{n_1+k_1,n_2+k_2}\right] , \tag{5.16}$$

where $C_{k_1k_2} = \sqrt{\frac{(k_1+k_2+1)!}{2\pi^2(k_1)!(k_2)!}}$. The BPS constraints at first order in $g_{\rm YM}$ are therefore given by setting the coefficients of all independent terms in (5.16) to zero. Thus the holomorphic constraints are parameterized by a pair of non-negative integers (n_1, n_2):

$$Q_{n_1n_2} \equiv \sum_{k_1,k_2=0}^{\infty} (k_1+k_2+2)\,\frac{C_{n_1n_2}C_{k_1k_2}}{C_{n_1+k_1,n_2+k_2}}\left[a^*_{k_1k_2},\, a_{n_1+k_1,n_2+k_2}\right] = 0 \ . \tag{5.17}$$

Since the expression (5.12) is explicitly self-adjoint under the $\star$ operation, the coefficients of the purely antiholomorphic terms $\left(\frac{\bar{Z}^1}{r^2}\right)^{n_1}\left(\frac{\bar{Z}^2}{r^2}\right)^{n_2}$ are simply the hermitian conjugates $(Q_{n_1n_2})^\dagger$.

We can interpret one of these constraints as the traditional Gauss law, which arises from the fact that the operator $\delta\bar{\delta}$ is a Laplacian:

$$\delta\bar{\delta} = r^2 \partial_I \partial_{\bar{I}} = \frac{1}{r}\partial_r r^3 \partial_r + \nabla_{S^3}\,, \tag{5.18}$$

and integration of the left hand side of (5.11) over a 3-sphere, with fixed r, vanishes. The S^3 part of the Laplacian obviously vanishes on integration because it is a total derivative. For the remaining radial derivative part, we recall that all functions appearing in our equations are degree 0 in $\bar{Z}$, which according to equation (F.4) means that

$$2\bar{Z}\cdot\bar{\partial} = r\partial_r - i\partial_\psi\ . \tag{5.19}$$

Since the radial derivative part of $\delta\bar{\delta}$ acts on a degree 0 function, we may make the replacement $\partial_r \overset{eff}{=} \frac{i}{r}\partial_\psi$, which makes it clear that the radial part of the Laplacian is also a total derivative. We then arrive at the Gauss Law consistency condition:

$$\int_{S^3} \left(\left[\bar{\delta} f_0^\star, \delta f_0\right] + 2\left[f_0^\star, f_0\right] \right) = 0\,, \tag{5.20}$$

which we recognize as:

$$Q_{00} = C_{00} \sum_{k_1,k_2=0}^{\infty} (k_1{+}k_2{+}2) \left[a^*_{k_1 k_2},\, a_{k_1 k_2}\right]\ . \tag{5.21}$$

To summarize, expanding f_0 in spherical harmonics $f_0 = \sum_{n_1,n_2=0}^{\infty} a_{n_1 n_2} Y_{n_1 n_2}(Z^1, Z^2)$, f_0 can be perturbed to a nearby BPS solution of a weakly interacting theory only if it solves the equations $Q_{n_1 n_2} = 0$ for all nonnegative n_1, n_2. We may consider this set of constraints to be a generalization of the Gauss law.

5.2 Perturbative expansion including scalars

Now we generalize the perturbative expansion of section 5.1 to include the scalar fields. The supersymmetric configurations of the free theory are parameterized by four unconstrained holomorphic functions: f_0 for the gauge field, and ϕ_0^i for three chiral scalars. Recall that the zeroth order value of the function g in the gauge potential, (4.28), is zero.

We now turn to the $\mathcal{O}(g_{\rm YM})$ analysis. We write the following set of unsolved BPS equations

$$F_{1\bar{1}} + F_{2\bar{2}} + \frac{i}{4}\left[\phi^i, \bar{\phi}^i\right] = 0 \tag{5.22}$$

$$\bar{\mathcal{D}} F_{12} + \frac{i}{4}\left[\phi^i, Z\cdot D\phi^{i\star}\right] - \frac{i}{4}\left[\phi^i, \phi^{i\star}\right] = 0 \tag{5.23}$$

$$D_{\bar{I}}\phi^i = 0\,, \qquad \left[\phi^i, \phi^j\right] = 0 \tag{5.24}$$

in terms of the functions f, g, ϕ^i as in section 5.1 and expand them in $g_{\rm YM}$. The BPS and Gauss equations at order $\mathcal{O}(g_{\rm YM})$ give equations for g_1, f_1 and ϕ_1^i with source terms given by f_0, ϕ_0^i. The equations are

$$g_1 = i(\bar{\delta} f_1 - \delta f_1^\star) - [f_0, f_0^\star] + \frac{1}{4}\left[\phi_0^i, {\phi_0^i}^\star\right] \tag{5.25}$$

$$\delta\bar{\delta}\bar{\delta}f_1 - \bar{\delta}\delta\delta f_1^\star = i\Big([\bar{\delta}f_0^\star, \delta f_0] + 2[f_0^\star, f_0]\Big) \qquad (5.26)$$
$$\times\frac{i}{4}\Big([\phi_0^{i\,\star}, \phi_0^i] + [\bar{\delta}\phi_0^{i\,\star}, \delta\phi_0^i] + (\delta\bar{\delta} + \bar{\delta}\delta)[\phi_0^i, \phi_0^{i\,\star}]\Big)$$
$$\bar{\delta}\phi_1^i = +i[f_0^\star, \phi_0^i] \qquad (5.27)$$
$$0 = [\phi_0^i, \phi_0^j]\ . \qquad (5.28)$$

In equations (5.25) and (5.26), there is an implicit sum over i on the right hand sides. The first equation specifies g_1 explicitly in terms of other fields, while the latter three equations require integrability conditions which generalize the constraints of section 5.1.

We will now obtain explicit expressions for the constraints. One class of constraints arises from equation (5.26) and generalizes the constraints we found in section 5.1. We will label these constraints as L_{mn} and the computation leading to their explicit expression may be found in appendix F.3. We now expand f_0 as in equation (5.13) and the scalars as $\phi^i = \sum_{n_1,n_2=0}^\infty b^i_{n_1n_2} Y_{n_1n_2}$. The result is that the holomorphic part of the right hand side of equation (5.26) is:

$$\sum_{n_1,n_2=0}^\infty Y_{n_1n_2} \sum_{k_1,k_2=0}^\infty c^{n_1n_2}_{k_1k_2} \Big((k_1+k_2+2)\left[a^*_{k_1k_2},\, a_{n_1+k_1,n_2+k_2}\right] + \frac{1}{4}(n_1+n_2+k_1+k_2+1)\left[b^{i*}_{k_1k_2},\, b^i_{n_1+k_1,n_2+k_2}\right]\Big) \qquad (5.29)$$

where the coefficient $c^{n_1n_2}_{k_1k_2}$ is defined following the notation of section 5.1 as:

$$c^{n_1n_2}_{k_1k_2} \equiv \frac{C_{n_1,n_2}C_{k_1,k_2}}{C_{n_1+k_1,n_2+k_2}}\ . \qquad (5.30)$$

The constraint arising from equation (5.27) is that the anti-holomorphic part of $[f_0^\star, \phi_0^i]$ should be zero:

$$\sum_{n_1,n_2=0}^\infty Y^\star_{n_1n_2} \sum_{k_1,k_2=0}^\infty c^{n_1n_2}_{k_1k_2}\left[a^*_{n_1+k_1,n_2+k_2},\, b^i_{k_1k_2}\right] = 0\ . \qquad (5.31)$$

The constraints corresponding to the vanishing of holomorphic parts are therefore

$$\begin{aligned} 0 = J^i_{n_1n_2} &\equiv \sum_{k_1,k_2=0}^\infty c^{n_1n_2}_{k_1k_2}\left[a^*_{n_1+k_1,n_2+k_2},\, b^i_{k_1k_2}\right] \\ 0 = L_{n_1n_2} &\equiv \sum_{k_1,k_2=0}^\infty c^{n_1n_2}_{k_1k_2}\Big((k_1+k_2+2)\left[a^*_{k_1k_2},\, a_{n_1+k_1,n_2+k_2}\right] \\ &\qquad + \frac{1}{4}(n_1+n_2+k_1+k_2+1)\left[b^{i*}_{k_1k_2},\, b^i_{n_1+k_1,n_2+k_2}\right]\Big) \end{aligned} \qquad (5.32)$$

$L_{n_1n_2} = 0$ arise from setting the purely holomorphic part of the source in (5.26) to zero, while $J^i_{n_1n_2} = 0$ arise from setting the purely anti-holomorphic part of the source in (5.27) to zero. Finally, equation (5.28) is itself a constraint which we will call

$$M^{ij} = \left[\phi_0^i, \phi_0^j\right] = 0\ . \qquad (5.33)$$

The BPS solutions of the free theory should satisfy $L_{n_1n_2} = J^i_{n_1n_2} = 0$ and $M^{ij} = 0$ to be lifted to a nearby BPS solutions of the weakly interacting theory.

6. Quantization of classically $\frac{1}{16}$ BPS solutions

We will now consider the quantization of these classical solutions. The quantization of classically BPS solutions has been considered before, often in the context of gravitational solutions [36 – 38], or probe branes [29, 20, 21], and more recently in the context of conformal field theory [35]. The space of all classical solutions to particular equations of motion may be identified with the classical phase space of the system. Such an identification is possible because each classical solution is determined by initial data which correspond to a specification of "position" coordinates and canonically conjugate momentum coordinates at the initial time. This provides a map between classical solutions of the equations of motion and points in phase space. We obtained constraints on the free BPS solutions which, when satisfied, allow the free solutions to be lifted to solutions at infinitesimal coupling so that a point in the phase space of the $\mathcal{N} = 4$ SYM 1/16 BPS sector is identified by specifying the holomorphic functions $f_0(Z_1, Z_2), \phi^i_0(Z_1, Z_2)$ in such a way that the constraints, $L_{n_1n_2} = J^i_{n_1n_2} = 0$ and $M^{ij} = 0$, are all satisfied.

6.1 Quantization prescription

Since the phase space above is described with the use of constraints, it will be most convenient to perform a constrained quantization, using the symplectic form of free field theory and attempting to include the constraints appropriately. Thus, we first consider the symplectic form of free $\mathcal{N} = 4$ Yang-Mills theory, evaluated on the space of 1/16 BPS solutions spanned by unconstrained $f_0(Z)$ and $\phi^i(Z)$.

The contribution of the $\mathrm{U}(N)$ gauge field to the symplectic form of the free theory is given by N^2 copies of a $\mathrm{U}(1)$ gauge field. A $\mathrm{U}(1)$ gauge field contributes a symplectic form

$$\omega = \int_{S^3} dF^{0i}_S \wedge dA^S_i = i \int_{S^3} d\left(x \cdot \partial A^\mu\right) \wedge dA_\mu \,, \tag{6.1}$$

where A^i_S is defined in (4.14).

For the free field theory BPS solution

$$A_I = \epsilon_{IJ} Z^J f_0 \,, \quad A_{\bar I} = \frac{\epsilon_{\bar I \bar J} \bar Z^J}{r^4} f_0^\star \,, \tag{6.2}$$

one obtains

$$\begin{aligned} x \cdot \partial A_I &= (Z\cdot\partial) A_I = \epsilon_{IJ} Z^J \left(Z\cdot\partial + 1\right) f_0 \\ x \cdot \partial A_{\bar I} &= (Z\cdot\partial - 1) A_{\bar I} = \frac{\epsilon_{\bar I \bar J}\bar Z^J}{r^4} (Z\cdot\partial - 3) f_0^\star \,. \end{aligned} \tag{6.3}$$

We plug this into the symplectic form (6.1) and find that the symplectic form evaluated on the 1/16 BPS solutions is

$$\begin{aligned} \omega &= 2i \int_{S^3} d\left(x\cdot\partial A_I\right) \wedge dA_{\bar I} + d\left(x \cdot \partial A_{\bar I}\right) \wedge dA_I \\ &= 2i \int_{S^3} 4 df_0 \wedge df_0^\star + d(Z\cdot\partial f_0) \wedge df_0^\star - df_0 \wedge d(Z\cdot\partial f_0^\star) \,. \end{aligned} \tag{6.4}$$

Expanding $f_0 = \sum_{n_1 n_2} a_{n_1 n_2} Y_{n_1 n_2}$ as in section 5.1, one obtains

$$\omega = 2i \sum_{n_1 n_2} da_{n_1 n_2} \wedge da^*_{n_1 n_2} \Big(4 + 2(n_1 + n_2)\Big) \ . \tag{6.5}$$

This means that

$$A^*_{n_1 n_2} \equiv 2\sqrt{n_1 + n_2 + 2}\ a_{n_1 n_2} \tag{6.6}$$

are normalized creation operators. When quantized, they will satisfy $[A_{m_1 m_2}{}^a{}_b, (A_{n_1 n_2}{}^c{}_d)^\dagger] = \delta_{m_1,n_1}\delta_{m_2,n_2}\delta^a_c\delta^d_b$.

The contribution of a scalar field to the symplectic form is

$$\begin{aligned} \omega &= \int_{S^3} d(\partial_0 {\phi_S}^*) \wedge d\phi_S + d(\partial_0 \phi_S) \wedge {\phi_S}^* \\ &= i \int_{S^3} 2d\phi \wedge d\phi^\star + d(Z \cdot \partial\phi) \wedge d\phi^\star - d\phi \wedge d(Z \cdot \partial\phi^\star)\,, \end{aligned} \tag{6.7}$$

where ϕ_S is a scalar on $S^3 \times \mathbb{R}$ and ϕ is a scalar field after a conformal transformation to $\mathbb{R}^4$, i.e. ϕ_S and ϕ are related as in equation (4.14). Expanding $\phi = \sum_{n_1,n_2} b_{n_1 n_2} Y_{n_1 n_2}$ as in previous sections, one obtains

$$2i \sum_{n_1,n_2=0}^{\infty} db_{n_1 n_2} \wedge db^*_{n_1 n_2}(n_1 + n_2 + 1)\ . \tag{6.8}$$

This implies that

$$B^*_{n_1 n_2} \equiv \sqrt{2(n_1 + n_2 + 1)}\ b_{n_1 n_2} \tag{6.9}$$

are normalized creation operators with standard commutation relations.

The symplectic form of free $\mathcal{N} = 4$ SYM in the bosonic sector is simply the sum of the contributions from the N^2 U(1) gauge fields and the 3 scalars. We will now consider the constraints of section 5.2, promoting the coefficients $A_{n_1 n_2}, A^*_{n_1 n_2}, B_{n_1 n_2}, B^*_{n_1 n_2}$ to creation/destruction operators.

First we will define more convenient creation and annihilation operators as:[8]

$$\begin{aligned} \alpha_{k_1 k_2} &= \sqrt{k_1 + k_2 + 2}\ C_{k_1 k_2}\ A_{k_1 k_2} & \alpha^\dagger_{k_1 k_2} &= \frac{1}{C_{k_1 k_2}\sqrt{k_1 + k_2 + 2}} A^*_{k_1 k_2} \\ \beta_{k_1 k_2} &= \frac{C_{k_1 k_2}}{\sqrt{k_1 + k_2 + 1}} B_{k_1 k_2} & \beta^\dagger_{k_1 k_2} &= \frac{\sqrt{k_1 + k_2 + 1}}{C_{k_1 k_2}} B^*_{k_1 k_2} \end{aligned} \tag{6.10}$$

These operators are convenient because they have standard commutation relations and they appear to be more natural variables for expressing the constraints. We will use a shorthand for the subscripts as $X_{k_1 k_2} \sim X_k$ to make equations more transparent. In terms

[8] Abusing the notation, we write $\alpha^\dagger_{k_1 k_2}$ and $\beta^\dagger_{k_1 k_2}$ even if they are not Hermitian conjugates of $\alpha_{k_1 k_2}$ and $\beta_{k_1 k_2}$ respectively.

of the new variables, the constraints in the classical theory become operators:

$$L_n = \frac{1}{2}\sum_{k=0}^{\infty}\left[\alpha_{n+k}, \alpha_k^{\dagger}\right] + \left[\beta^i_{n+k}, \beta_k^{i\dagger}\right] \tag{6.11}$$

$$J^i_n = \sum_{k=0}^{\infty}\left[\alpha_{n+k}, \beta_k^{i\dagger}\right] \tag{6.12}$$

$$M^{ij}_n = \sum_{k=0}^{n}\left({}^{k_1+k_2}C_{k_1}\,{}^{n_1+n_2-k_1-k_2}C_{n_1-k_1}\right)\left[\beta^i_k, \beta^j_{n-k}\right] \tag{6.13}$$

up to overall multiplicative constants, which we drop. To resolve the normal-ordering ambiguity that occurs for L_0, we define L_0 to be the normal-ordered one, having annihilation operators on the right side of the creation operators. ${}^{k_1+k_2}C_{k_1}$ is the binomial coefficient k_1+k_2 choose k_1 and we have suppressed the $\mathrm{U}(N)$ matrix indices.

Now we must consider how to impose the constraints correctly. Classically, $L_k, L_l^{\dagger}, J^i_m, J^{i\dagger}_n, M^{ij}_p, M^{ij\dagger}_q$ are all constrained to be zero. We will not perform a systematic quantization of these first class constraints, rather, we adopt an approach similar to the Old Covariant Quantization (OCQ). Since we will find that our prescription results in the same cohomology as described in section 2.3, we presume that a rigorous quantization will lead to the same result.

As in OCQ, we will first quantize the 1/16 BPS sector without considering the dynamical constraints, (6.11)–(6.13), and then impose the constraints as operator relations in Hilbert space. Explicitly, we define a vacuum $|0\rangle$ that satisfies $\alpha_k|0\rangle = \beta^i_k|0\rangle = 0$ for all k and let the $\alpha_k^{\dagger}, \beta_k^{i\dagger}$ operate on $|0\rangle$ to produce an unconstrained Hilbert space. This unconstrained Hilbert space corresponds simply to the 1/16 BPS sector of the free $\mathcal{N}=4$ SYM theory.

Now we impose the constraints in this Hilbert space. Following OCQ terminology, we will call any state which remains BPS at infinitesimal coupling "physical" and require such a state, $|\psi\rangle$, to satisfy:

$$L_k|\psi\rangle = J^i_m|\psi\rangle = M^{ij}_p|\psi\rangle = 0. \tag{6.14}$$

This ensures that the matrix elements of all the constraints are zero between any two physical states. For example, if $|\psi\rangle, |\chi\rangle$ are physical states, then we have:

$$\langle\psi|L_k^{\dagger}|\chi\rangle = \langle L_k\psi|\chi\rangle = 0\ . \tag{6.15}$$

One might wonder whether we should require L_n or $L_n^{\dagger}$ to annihilate physical states, since either condition would be sufficient to set all matrix elements of all constraints to zero. Although being a bit *ad hoc*, it is clear that we should require L_n and not $L_n^{\dagger}$ to annihilate physical states. This is because we want states made of symmetrized gauge invariant scalar zero modes $(\beta^i_{00})^{\dagger}$, which belong to the $\frac{1}{8}$ BPS Hilbert space, to be physical. If we set $L_n|\Psi\rangle = 0$, where $|\Psi\rangle$ is such a $\frac{1}{8}$ BPS state, then the only term of L_n which acts nontrivially on $|\Psi\rangle$ is $\sum_{i=1}^{3}[(\beta^i_{00})^{\dagger}, \beta^i_{00}]$ in L_{00}, which simply requires $\mathrm{U}(N)$ gauge invariance. If we set $L_n^{\dagger}|\Psi\rangle = 0$, however, terms of the form $[\beta^i_{00}, (\beta^i_{n_1 n_2})^{\dagger}]$ in $(L_{n_1 n_2})^{\dagger}$ act nontrivially and these $\frac{1}{8}$ BPS states would not be physical.

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As a simple illustration and test of the above prescription, we will try to identify a single trace physical state made of a single scalar only. We start from

$$|2,2\rangle \equiv \mathrm{tr}\left(b_1^\dagger b_1^\dagger + 2b_0^\dagger b_2^\dagger\right)|0\rangle\,, \tag{6.16}$$

where $b_n^\dagger \equiv {\beta^3_{n0}}^\dagger$ is a creation operator for the scalar ϕ^3, associated with $(Z^1)^n$. This state maps by the state operator map to[9] (following the conventions used in section 2):

$$(\partial_{+\dot{+}})^2\mathrm{tr}(\bar{\phi}^3)^2 = 2\mathrm{tr}\left(\,(D_{+\dot{+}}\bar{\phi}^3)^2 + \bar{\phi}^3 D^2_{+\dot{+}}\bar{\phi}^3\right)\,. \tag{6.17}$$

which according to section 2.3 is an allowed 1/16 BPS state at infinitesimal coupling.

We will now check that $|2,2\rangle$ is annihilated by L_n. It is clear that only the L_{n0} constraints need to be checked. $L_{n0}|2,2\rangle = 0$ for $n \geq 3$ because the state does not involve a creation $b_n^\dagger$ with $n \geq 3$ and $L_{00}|2,2\rangle = 0$ is automatic because the trace guarantees gauge-invariance. Finally, from $[(b_m)^a{}_b, (b_n^\dagger)^c{}_d] = \delta_{mn}\delta^a_d\delta^c_b$, we find

$$(L_{20})^a{}_b|2,2\rangle = 2\left(\,\left(b_0^\dagger\right)^a{}_c(b_2)^c{}_b - \left(b_0^\dagger\right)^c{}_b(b_2)^a{}_c\right)\mathrm{tr}\left(b_0^\dagger b_2^\dagger\right)|0\rangle = 2\left[b_0^\dagger, b_0^\dagger\right]^a{}_b|0\rangle = 0\,, \tag{6.18}$$

and

$$(L_{10})^a{}_b|2,2\rangle = \left[b_0^\dagger, b_1\right]^a{}_b\mathrm{tr}\left(b_1^\dagger b_1^\dagger\right)|0\rangle + 2\left[b_1^\dagger, b_2\right]^a{}_b\mathrm{tr}\left(b_0^\dagger b_2^\dagger\right)|0\rangle = 2\left(\left[b_0^\dagger, b_1^\dagger\right] + \left[b_1^\dagger, b_0^\dagger\right]\right)|0\rangle = 0 \tag{6.19}$$

So that $|2,2\rangle$ is a physical state if we require that physical states be annihilated by L_n as discussed above.

When considering the J^i_m constraint, one may again wonder whether we should require J^i_m or $(J^i_m)^\dagger$ to annihilate physical states. We again use the $\frac{1}{8}$ BPS primaries as a guide. The $J^i_{n_1n_2}$ operators automatically annihilate the above $\frac{1}{8}$ BPS primaries because of the gauge field annihilation operators $\alpha_{n_1+k_1,n_2+k_2}$ appearing in every term of $J^i_{n_1n_2}$, but the $(J^i_{n_1n_2})^\dagger$ operators do not annihilate these states. Therefore we must require that physical states satisfy $J^i_n|\psi\rangle = 0$.

It is clear that we must impose $M^{ij}_m|\psi\rangle = 0$ because $(M^{ij}_m)^\dagger$ contains only creation operators and therefore cannot annihilate the $\frac{1}{8}$ BPS primaries. On the other hand, M^{ij}_m does annihilate these states because it annihilates states which are symmetrized on the SU(3) index.

Finally, our quantization prescription is that we impose constraints on the free 1/16 BPS Hilbert space: States that remain 1/16 BPS at infinitesimal coupling, (physical states), should be annihilated by L_n, J^i_n, M^{ij}_n.

6.2 Relation between classically derived constraints and the analysis of section 2.3

We will now map the results of the classical analysis to equation (2.9). We collect the oscillators $\alpha_{k_1k_2}, \beta^i_{k_1k_2}$, which resulted from the quantization of the classical solutions (6.10),

[9]Note that under the state operator map, the operator $D^n_{+\dot{+}}\bar{\phi}^3$ maps to the state $n!\beta^3_{n0}$

into fields similar to those defined above, which will allow us to consider the BPS constraints obtained in the classical analysis in more detail:[10]

$$\begin{aligned}\beta^i(\bar z) &= \sum_{k_1k_2}\frac{(k_1+k_2+1)!}{k_1!k_2!}\beta^i_{k_1k_2}\bar z_1^{k_1}\bar z_2^{k_2}\,, &\quad \beta^{i\dagger}(z) &= \sum_{k_1k_2}\beta^{i\dagger}_{k_1k_2}z_1^{k_1}z_2^{k_2}\\ \alpha(\bar z) &= \sum_{k_1k_2}\frac{(k_1+k_2+1)!}{k_1!k_2!}\alpha_{k_1k_2}\bar z_1^{k_1}\bar z_2^{k_2}\,, &\quad \alpha^{\dagger}(z) &= \sum_{k_1k_2}\alpha^{\dagger}_{k_1k_2}z_1^{k_1}z_2^{k_2}\,,\end{aligned} \tag{6.20}$$

We have defined these fields such that holomorphic fields are creation operators and anti-holomorphic fields are destruction operators. With these definitions, we can write the constraints obtained in the classical analysis of section 6 as:

$$\begin{aligned}L_{W_L} &= \frac{1}{2}\int_{S^3} Tr\ W_L(z)\left[\alpha(\bar z),\alpha^\dagger(z)\right] + W_L(z)\left[\beta^i(\bar z),\beta^{i\dagger}(z)\right]\\ J^i_{W_J} &= \frac{1}{2}\int_{S^3} Tr\ W_J(z)\left[\alpha(\bar z),\beta^{i\dagger}(z)\right]\\ M^{ij}_{W_M} &= \frac{1}{2}\int_{S^3} Tr\ W_M(z)\left[\frac{1}{1+\bar z\cdot\bar\partial}\beta^i(\bar z),\frac{1}{1+\bar z\cdot\bar\partial}\beta^j(\bar z)\right]\,,\end{aligned} \tag{6.21}$$

where the integrals are over the unit sphere. The constraints are parameterized by the matrices $W_L(z), W_J(z), W_M(z)$. We recover the constraints as listed previously by:

$$L_n = L_{z^n}\,,\quad J^i_n = J^i_{z^n}\,,\quad M^{ij}_n = M^{ij}_{z^n}\ . \tag{6.22}$$

The notation here means that to recover $L_{n_1n_2}$ for example, we let $W_L = z_1^{n_1}z_2^{n_2}$. The commutation relations of the local fields are for the scalars, for example:

$$[\beta(\bar z),\beta^\dagger(w)] = (1+\bar z\cdot\bar\partial)\frac{1}{1-\bar z_1w_1-\bar z_2w_2} = \frac{1}{(1-\bar z_iw_i)^2} \tag{6.23}$$

The right hand side of this commutation relation acts like a delta function for integration of holomorphic functions on S^3. That is:

$$\int_{S^3}\frac{f(z)}{(1-\bar z_iw_i)^2} = \Omega_3 f(w) \tag{6.24}$$

This is a type of cauchy integral formula in 2 complex dimensions.

Now we list the action of the constraints on the fields $\alpha^\dagger(z),\beta^{i\dagger}(z)$.

$$\begin{aligned}[L_{W_L},(\beta^{i\dagger}(z))^{pq}] &= [\beta^{i\dagger}(z),W_L(z)]^{pq}\\ [L_{W_L},(\alpha^{\dagger}(z))^{pq}] &= [\alpha^{\dagger}(z),W_L(z)]^{pq}\\ [J^i_{W_J},(\alpha^{\dagger}(z))^{pq}] &= [\beta^{i\dagger}(z),W_J(z)]^{pq}\end{aligned} \tag{6.25}$$

the lower i index here is the SU(3) index of the scalar and p, q are gauge indices.

[10]Recall that we are abusing notation: $\alpha_{k_1k_2},\alpha^\dagger_{k_1k_2}$ are creation and destruction operators with standard commutation relations, but $\alpha^\dagger_{k_1k_2}$ is not really the hermitian conjugate of $\alpha_{k_1k_2}$

Further we have:

$$\left[\frac{1}{1+\bar{z}\cdot\bar{\partial}}(\beta^i(\bar{z}))^{mn},(\beta^{j\dagger}(w))^{pq}\right]=\frac{\delta^{ij}\delta^{mq}\delta^{np}}{1-\bar{z}^a w^a}$$

Defining $\Phi(\bar{z})=\frac{1}{1+\bar{z}\cdot\bar{\partial}}\beta(\bar{z})$, this gives:

$$\left[M^{ij}_{W_M},(\beta^\dagger_k)^{pq}(w)\right]=\int_{S^3}\left\{[\Phi^j(\bar{z}),W_M(z)]^{pq}\delta^{ik}-[\Phi^i(\bar{z}),W_M(z)]^{pq}\delta^{jk}\right\}\frac{1}{1-\bar{z}\cdot w} \tag{6.26}$$

The right hand side still contains annihilation operators $\Phi(\bar{z})$ so M naturally acts on pairs of creation fields $\beta^{k\dagger}(z)\beta^{l\dagger}(z)$ and we obtain:

$$[M^{ij}_{W_M},(\beta^{k\dagger}(w))^{pq}(\beta^{l\dagger}(w))^{st}]=(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk})\left(W^{sq}_M(w)\delta^{pt}-W^{pt}_M(w)\delta^{sq}\right)\,. \tag{6.27}$$

We have assumed here that the operator $(\beta^{k\dagger}(w))^{pq}(\beta^{l\dagger}(w))^{st}$ is acting on the vacuum, that is it stands for $(\beta^{k\dagger}(w))^{pq}(\beta^{l\dagger}(w))^{st}|0\rangle$. We therefore dropped all terms containing destruction operators. The factor $(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk})$ in equation (6.27) indicates that the $M^{ij}_{W_M}$ constraint requires 1/16 BPS operators at infinitesimal coupling to be symmetrized on the SU(3) index. In fact, similar factors which are antisymmetric in the SU(3) indices, appear when $M^{ij}_{W_M}$ acts on arbitrary single-trace states so that $M^{ij}_{W_M}$ requires symmetrization of SU(3) indices inside traces quite generally.

With these considerations in hand, we now consider the quantization prescription obtained from the classical analysis of section 6 in cohomological terms. The L_{W_L} constraint acts on the fields $\alpha^\dagger(z),\beta^{i\dagger}(z)$ as a gauge transformation and requires all 1/16 BPS operators at infinitesimal coupling to be traces of the fields $\alpha^\dagger(z),\beta^{i\dagger}(z)$, where all fields inside a given trace are at the same position. Therefore it is clear that L_{W_L} implements the same gauge invariance constraint discussed in section 2.3, with the parameter, W_L, of the gauge transformation mapping to a derivative of the gaugino, $W_L(z)\sim z^{\dot\alpha}(1+z\cdot\partial)^{-1}\bar\lambda_{\dot\alpha}(z)$, in the cohomology picture. The J_{W_J} constraint arises in equation (2.9) from the action of Q^1_- on $f(z)$ which produces $[\psi_{n+}(z),\bar\phi^n(z)]$. The parameter W_J maps to the chiralino $W_J(z)\sim\psi_{n+}(z)$ in the cohomology language. Finally, the $M^{ij}_{W_M}$ constraints correspond to the action of the superconformal generator S^-_4 on commutators of scalars. In section 2.3 we worked in terms of cohomology and this superconformal constraint from S^-_4 was interpreted as the exactness condition that arises because $[\bar\phi^n(z),\bar\phi^p(z)]$ appears on the right hand side of (2.9).

Now it is clear that the analysis of the classical configurations and their quantization agree with the rules obtained in section 2.3.

7. Concluding remarks

In this paper we studied the $\frac{1}{16}$-BPS states of the weakly-coupled $\mathcal{N}=4$ Yang-Mills theory. We formulated the problem in terms of local fields which generate covariant derivatives acting on fields. In particular, we thoroughly investigated the $\frac{1}{16}$-BPS cohomology made of bosonic letters. We obtained the exact partition function in special limits, and also an upper bound partition function which turns out to be useful. We also gave a physical interpretation of the local fields by studying a set of classical BPS equations for $\frac{1}{16}$-BPS configurations in the bosonic sector, and suitably quantizing them.

The classical cohomology we consider is in 1-to-1 correspondence with the zero eigenstates of the 1-loop Hamiltonian acting on the Hilbert space of $\frac{1}{16}$-BPS states in the free Yang-Mills theory. In this sense, our analysis generalizes and strengthens the result of [22] to the finite N case in certain subsectors or energy regime.

Using our exact partition function for scalars derived in special limits, we have also shown that a dual description in the strong coupling regime, in terms of giant gravitons, sometimes predicts exactly the same result as ours. We also pointed out that our exact result shows qualitatively similar behaviors to the result obtained from a naive quantization of dual giant gravitons. The error of the latter result is argued to be due to the intersection of multiple dual giant gravitons. As we commented in section 3.4, it is interesting to see if one can correctly quantize them and obtain the exact partition function.

We used our upper bound partition function to conclusively argue that the asymptotic degeneracy with large charges is not big enough to form a supersymmetric black hole.

The most important problem which follows our analysis, combined with the conjecture that weakly coupled Yang-Mills theory captures the exact supersymmetric spectrum, should be to find the large number of $\frac{1}{16}$-BPS states in the high energy regime, with their entropy scaling like N^2 if $E \sim N^2$, and furthermore to identify it with the entropy of the known supersymmetric black holes [11]. Since our analysis in the bosonic sector was quite comprehensive, perhaps fermionic letters should play important roles. See [18] for some attempts in this direction.

A more modest question along the similar direction would be to understand the finite N cohomology that we investigated in this paper, or any generalization thereof, in the strongly-coupled regime in terms of giant gravitons. Even if the giant graviton provides an effective description of BPS states in the energy regime around $E \sim N$, it hopefully could also give us some new insights or clue towards understanding the even higher energy regime. For the latter purpose, one probably would have to go beyond the solutions of [33] by exciting other degrees of freedom (see [39] also).

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A. Action of supercharges on the letters of $\mathcal{N} = 4$ super Yang-Mills

The SU(4) vector Q^i_α with $i = 1, \ldots, 4$ has been divided into a special supercharge Q_α and an SU(3) vector Q^m_α with $m = 1, \ldots, 3$. In terms of the SU(4) notation, Q_α is the 4th component and Q^m_α correspond to the $1, 2, 3$ components of the SU(4) vector. The indices m, n, p below run from $1, \ldots, 3$ and the indices $\alpha, \beta, \gamma, \delta$ run over $1, 2$. The transformation rule of $\mathcal{N} = 4$ Yang-Mills theory, say in the appendix of [20], is decomposed as

$$
\begin{aligned}
[Q^m_\alpha, \bar{\phi}^n] &= -\epsilon^{mnp}\psi_{p\alpha} \\
[Q_\alpha, \bar{\phi}^n] &= 0 \\
[Q^m_\alpha, \phi_p] &= -\delta^m_p \lambda_\alpha \\
[Q_\alpha, \phi_p] &= \psi_{p\alpha} \\
\{Q^m_\alpha, \lambda_\beta\} &= \epsilon_{\alpha\beta}\epsilon^{mnp}[\phi_n, \phi_p] \\
\{Q_\alpha, \lambda_\beta\} &= 2if_{\alpha\beta} + \epsilon_{\alpha\beta}[\phi_k, \bar{\phi}^k] \\
\{Q^m_\alpha, \psi_{n\beta}\} &= 2i\delta^m_n f_{\alpha\beta} - 2\epsilon_{\alpha\beta}[\bar{\phi}^m, \phi_n] + \epsilon_{\alpha\beta}\delta^m_n[\bar{\phi}^k, \phi_k] \\
\{Q_\alpha, \psi_{n\beta}\} &= -\epsilon_{\alpha\beta}\epsilon_{nmp}[\bar{\phi}^m, \bar{\phi}^p] \\
\{Q^m_\alpha, \bar{\lambda}_{\dot{\beta}}\} &= 2iD_{\alpha\dot{\beta}}\bar{\phi}^m \\
\{Q_\alpha, \bar{\lambda}_{\dot{\beta}}\} &= 0 \\
\{Q^m_\alpha, \bar{\psi}^n_{\dot{\beta}}\} &= -2i\epsilon^{mnp}D_{\alpha\dot{\beta}}\phi_p \\
\{Q_\alpha, \bar{\psi}^n_{\dot{\beta}}\} &= -2iD_{\alpha\dot{\beta}}\bar{\phi}^n \\
[Q^m_\alpha, A_{\beta\dot{\gamma}}] &= -\epsilon_{\alpha\beta}\bar{\psi}^m_{\dot{\gamma}} \\
[Q_\alpha, A_{\beta\dot{\gamma}}] &= -\epsilon_{\alpha\beta}\bar{\lambda}_{\dot{\gamma}} \\
[Q^m_\alpha, f_{\beta\gamma}] &= \epsilon_{\alpha\{\gamma}D^{\dot{\delta}}_{\beta\}}\bar{\psi}^m_{\dot{\delta}} \\
[Q_\alpha, f_{\beta\gamma}] &= \epsilon_{\alpha\{\gamma}D^{\dot{\delta}}_{\beta\}}\bar{\lambda}_{\dot{\delta}} = -\frac{i}{2}\epsilon_{\alpha\gamma}[\bar{\phi}^m, \psi_{m\beta}] - \frac{i}{2}\epsilon_{\alpha\beta}[\bar{\phi}^m, \psi_{m\gamma}] \\
[Q^m_\alpha, f_{\dot{\beta}\dot{\gamma}}] &= -D_{\alpha\{\dot{\beta}}\bar{\psi}^m_{\dot{\gamma}\}} \\
[Q_\alpha, f_{\dot{\beta}\dot{\gamma}}] &= -D_{\alpha\{\dot{\beta}}\bar{\lambda}_{\dot{\gamma}\}}
\end{aligned}
\tag{A.1}
$$

In the equation for $[Q_\alpha, f_{\beta\gamma}]$, we have used the fermionic equation of motion:

$$
\begin{aligned}
D_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}} &= i[\bar{\phi}^m, \psi_{m\alpha}] \\
D_{\beta\dot{\alpha}}\lambda^\beta &= i[\phi_m, \bar{\psi}^m_{\dot{\alpha}}]
\end{aligned}
\tag{A.2}
$$

The action of the supersymmetries $\bar{Q}^i_{\dot{\alpha}}$ with $i = 1, \ldots, 4$ are given by taking the simple complex conjugate of the above relations. With these definitions, the algebra satisfies $\{Q^i_\alpha, \bar{Q}_{j\dot{\beta}}\}X = 2i\delta^i_j D_{\alpha\dot{\beta}}X$ up to a gauge transformation.

B. Checks of the SU(2) partition function

We will now check the $N = 2$ partition function of section 3.2 explicitly. It suffices to check the SU(2) gauge group instead of U(2), since the contribution from the overcall U(1) is always factored out as $Z_1(\mu_i, \theta_a)$ as is also true in (3.2). Here we only record the check for the operators that contain a single instance of a single derivative. We also checked the operators involving two derivatives of the same kind (i.e. checks at the next order in angular momentum charge) from an analysis similar to this section (more involved), and also find agreement with the partition function of section 3.2. The SU(2) partition function that we would like to check is

$$Z_2(\mu_1,\mu_2,\theta)=\prod_{k=0}^{\infty}\frac{1-\mu_1^2\mu_2^2\theta^k}{(1-\mu_1^2\theta^k)(1-\mu_2^2\theta^k)(1-\mu_1\mu_2\theta^k)}\ . \tag{B.1}$$

The first two factors in the denominator of the U(2) partition function (3.16) are omitted, since, in SU(2), we do not have the generators $\partial^k\mathrm{tr}(X)$ and $\partial^k\mathrm{tr}(Y)$. The relation or syzygy 3.7 reduces to

$$\mathrm{tr}\left(X^2\right)\mathrm{tr}\left(Y^2\right)=\Big(\mathrm{tr}(XY)\Big)^2\ , \tag{B.2}$$

and the numerator of the partition function remains unchanged. The Taylor expansion of the above partition function in terms of θ is

$$\begin{aligned} Z_2=&\frac{1+\mu_1\mu_2}{(1-\mu_1^2)(1-\mu_2^2)}\Big\{1+\theta\left(\mu_1^2+\mu_2^2+\mu_1\mu_2-\mu_1^2\mu_2^2\right) \\ &+\theta^2\big(\mu_1^4+\mu_2^4+\mu_1^2\mu_2^2+\mu_1^3\mu_2+\mu_1\mu_2^3-\mu_1^4\mu_2^2-\mu_1^2\mu_2^4-\mu_1^3\mu_2^3+\mu_1^2+\mu_2^2+\mu_1\mu_2\big)+\mathcal{O}(\theta^3)\Big\}. \end{aligned} \tag{B.3}$$

To test the above partition function by an independent computation, we first take the scalars X and Y to be diagonal matrices

$$X=\begin{pmatrix} x & 0\\ 0 & -x\end{pmatrix}\ ,\quad Y=\begin{pmatrix} y & 0\\ 0 & -y\end{pmatrix}\ . \tag{B.4}$$

The relation is obviously satisfied. The gauge invariants are

$$\mathrm{tr}(X^2)\sim x^2\ ,\quad \mathrm{tr}(Y^2)\sim y^2\ ,\quad \mathrm{tr}(XY)\sim xy\ . \tag{B.5}$$

A gauge-invariant operator made of these three units must have the total number of x and y to be even, that is, either

$$x^{2m+2}y^{2n+2}\quad\text{or}\quad x^{2m+1}y^{2n+1}\qquad (m,n=0,1,2\cdots)\ . \tag{B.6}$$

Furthermore, for an operator with derivatives to be gauge-invariant, the derivatives ∂ should effectively act on these units only.

Let us check the sector with a single occurence of the derivative, that is, the coefficient of θ in $Z_2(\mu_1,\mu_2,\theta)$. The following set of operators

$$\begin{aligned} &(\partial x)x^{2m+1}y^{2n}\ ,\quad (\partial x)x^{2m+2}y^{2n+1} \\ &(\partial y)x^{2m}y^{2n+1}\ ,\quad (\partial y)x^{2m+1}y^{2n+2}\qquad (m,n=0,1,2\cdots) \end{aligned}$$

can be understood as containing the letter $\partial(x^2)$ or $\partial(y^2)$, and containing even number of x's and y's. The only other possible cases are those containing $\partial(xy)$, and which cannot be written as one of the above four operators. They are

$$\partial(xy)x^{2m}\ ,\quad \partial(xy)y^{2n}\qquad (m,n=0,1,2\cdots\ ;\ (0,0)\ \text{overcounted}) \tag{B.7}$$

since $(\partial x)y^{2n+1}$ and $(\partial y)x^{2m+1}$ terms in the above cannot be rewritten to contain $\partial(x^2)$ or $\partial(y^2)$ factors. The partition function for the above operators, apart from the common θ factor, is

$$\begin{aligned} Z_2(\mu_1,\mu_2,\theta)\Big|_{\theta^1} &= \frac{\mu_2^2+\mu_2^2+\mu_1^3\mu_2+\mu_1\mu_2^3}{(1-\mu_1^2)(1-\mu_2^2)}+\frac{\mu_1\mu_2}{1-\mu_1^2}+\frac{\mu_1\mu_2}{1-\mu_2^2}-\mu_1\mu_2 \\ &= \frac{(1+\mu_1\mu_2)(\mu_1^2+\mu_2^2+\mu_1\mu_2-\mu_1^2\mu_2^2)}{(1-\mu_1^2)(1-\mu_2^2)}\,. \end{aligned} \tag{B.8}$$

This agrees with the coefficient of θ in the proposed partition function.

There is another way to derive (B.1) based on the Koszul complex (see for example [40]). One introduces a local *anticommuting* operator C and a differential Δ such that

$$\begin{aligned} \Delta C &= \mathrm{tr}\left(X^2\right)\mathrm{tr}\left(Y^2\right)-\Big(\mathrm{tr}(XY)\Big)^2\,, \\ \Delta\mathrm{tr}\left(X^2\right) &= 0\,,\qquad \Delta\mathrm{tr}\left(Y^2\right)=0\,,\qquad \Delta\mathrm{tr}(XY)=0\,. \end{aligned} \tag{B.9}$$

The operators $\mathrm{tr}\left(X^2\right), \mathrm{tr}\left(Y^2\right)$ and $\mathrm{tr}\left(XY\right)$ are considered unrelated and the cohomology of the Koszul differential $H(\Delta)$ reproduces the space of constrained operators. The partition function Z_2 is now easily computed by taken into account the three free operators $\mathrm{tr}\left(X^2\right), \mathrm{tr}\left(Y^2\right), \mathrm{tr}\left(XY\right)$, which are bosonic, and the fermionic operator C. The latter scales as $\mu_1^2\mu_2^2$ if we scale X and Y with μ_1 and μ_2. So, combining these data, one obtains the formula (B.1): the three bosonic fields and their derivatives yield the contributions $(1-\mu_1^2\theta^k)(1-\mu_2^2\theta^k)(1-\mu_1\mu_2\theta^k)$ in the denominator of Z_2, while the anticommuting C leads to the contribution $(1-\mu_1^2\mu_2^2\theta^k)$ in the numerator.

C. Appendix to section 4

C.1 Derivation of equation (4.6)

In this appendix, we derive equation (4.6). The energy density (4.2) can be rearranged as:

$$\begin{aligned} \mathcal{E} =& \frac{1}{2}\left(F_{12}+s_1\frac{1}{2}[\phi^i,\bar\phi^i]\right)^2 - s_1\mathrm{tr}\left(\,F_{12}[\phi^i,\bar\phi^i]\right) \\ &+\frac{1}{2}\left(\,F_{a0}-s_2F_{a3}\right)^2+s_2F_{a0}F_{a3} \\ &+\frac{1}{2}\left|\,D_0\phi^i-s_2D_3\phi^i+s_3i\phi^i\right|^2+s_2\frac{1}{2}\left(D_0\phi^iD_3\bar\phi^i+D_0\bar\phi^iD_3\phi^i\right) \\ &-s_3\frac{i}{2}\left(\phi^iD_0\bar\phi^i-\bar\phi^iD_0\phi^i\right)+s_2s_3\frac{i}{2}\left(\phi^iD_3\bar\phi^i-\bar\phi^iD_3\phi^i\right) \\ &+\frac{1}{2}\left|\,(D_1+s_1iD_2)\phi^i\right|^2+is_1\left(D_1\phi^iD_2\bar\phi^i-D_2\phi^iD_1\bar\phi^i\right) \\ &+\frac{1}{2}(F_{03})^2+\frac{1}{4}\left|\,[\phi^i,\phi^j]\right|^2 \end{aligned} \tag{C.1}$$

where $s_{1,2,3}$ are $\pm$ signs and the trace on the gauge indices is understood.

We now note that

$$\int_{S^3}\mathrm{tr}\,D_1\phi^iD_2\bar\phi^i-\mathrm{tr}\,D_2\phi^iD_1\bar\phi^i=\int\mathrm{tr}\,\bar\phi^i[D_1,D_2]\phi^i-\epsilon_{ab}\partial_a\mathrm{tr}\left(\bar\phi^iD_b\phi^i\right)\;. \tag{C.2}$$

The second term is

$$\int_{S^3} \epsilon_{ab} E_a^\mu \partial_\mu \mathrm{tr}\left(E_b^\nu \bar{\phi}^i D_\nu \phi^i\right) = \int (\det e)\Big(\epsilon_{ab} E_a^\mu (\partial_\mu E_b^\nu) \mathrm{tr}(\bar{\phi}^i D_\nu \phi^i) + \epsilon_{ab} E_a^\mu E_b^\nu \partial_\mu \mathrm{tr}\left(\bar{\phi}^i D_\nu \phi^i\right)\Big)$$
$$= \int (\det e)[\partial_1, \partial_2]^\nu \mathrm{tr}(\bar{\phi}^i D_\nu \phi^i) + \epsilon^{\mu\nu\rho} e_\rho^3 \partial_\mu \mathrm{tr}\left(\bar{\phi}^i D_\nu \phi^i\right) , \tag{C.3}$$

where $[\,,\,]^\nu$ denotes the Lie commutator of directional derivatives and E_b^ν are components of the basis vectors of our local orthonormal frame on the S^3. That is E_b^ν are the components of the dual vectors to the left-invariant one forms listed in equation (4.5). Note that, up to a total derivative, the derivative acting on $\mathrm{tr}\, \bar{\phi}^i D_\nu \phi^i$ in the last term can effectively be regarded as acting on

$$\mathrm{tr}\, \bar{\phi}^i D_\nu \phi^i \overset{eff}{=} \frac{1}{2}\left(\mathrm{tr}\, \bar{\phi}^i D_\nu \phi^i - \mathrm{tr}\, D_\nu \bar{\phi}^i \phi^i\right) . \tag{C.4}$$

Combining these with the first term of equation (C.2), which is

$$\int \mathrm{tr}\, \bar{\phi}^i [D_1, D_2] \phi^i = \int -i \mathrm{tr}(\bar{\phi}^i [F_{12}, \phi^i]) + [\partial_1, \partial_2]^\mu \,\mathrm{tr}\, \bar{\phi}^i D_\mu \phi^i , \tag{C.5}$$

one obtains

$$\int_{S^3} \mathrm{tr}\Big(D_1 \phi^i D_2 \bar{\phi}^i - D_2 \phi^i D_1 \bar{\phi}^i\Big) = \tag{C.6}$$
$$= \int -i(\det e) \mathrm{tr}\left(\bar{\phi}^i [F_{12}, \phi^i]\right) - \frac{1}{2} \epsilon^{\mu\nu\rho} e_\rho^3 \partial_\mu \mathrm{tr}\left(\bar{\phi}^i D_\nu \phi^i - D_\nu \bar{\phi}^i \phi^i\right) .$$

Note that the directional derivative $\partial_3 = \frac{\partial}{\partial \psi}$ generates an isometry. If $\rho = \psi$, the second term is a total derivative which can be ignored since $e_\psi^3 = 1$. If $\mu = \psi$, it again becomes a total derivative because $e_\theta^3 = 0$ and e_ϕ^3 is independent of ψ. Therefore, the only term which gives nontrivial contribution is the term where $\nu = \psi$. So one can rearrange the second term of equation (C.6) as

$$+\int \frac{1}{2} \epsilon^{\mu\rho\psi} e_\rho^3 \partial_\mu \Big(\cdots\Big)_\nu = -\int \frac{1}{2} \epsilon^{\mu\rho\psi} (de^3)_{\mu\rho} \Big(\cdots\Big)_\nu = +\frac{1}{2} \int (\det e) \mathrm{tr}\Big(\bar{\phi}^i D_\nu \phi^i - D_\nu \bar{\phi}^i \phi^i\Big). \tag{C.7}$$

Therefore, we finally obtain

$$i \int_{S^3} \mathrm{tr}\, D_1 \phi^i D_2 \bar{\phi}^i - \mathrm{tr}\, D_2 \phi^i D_1 \bar{\phi}^i = \int (\det e) \mathrm{tr}\left(F_{12} [\phi^i, \bar{\phi}^i] + \frac{i}{2}\left(\bar{\phi}^i D_\nu \phi^i - D_\nu \bar{\phi}^i \phi^i\right)\right) . \tag{C.8}$$

Inserting this result into the 5th line of the complete-squared energy functional (C.1), one finds that the first term is canceled against the same term in the first line of (C.1). Furthermore, if $s_1 = +s_2 s_3$, the second term is canceled with a cross term in the 4th line of (C.1). The result is

$$\begin{aligned} \mathcal{E} = \mathrm{tr} \Bigg[& \frac{1}{2}\left(F_{12} + s_1 \frac{1}{2} [\phi^i, \bar{\phi}^i]\right)^2 + \frac{1}{2}\Big(F_{a0} - s_2 F_{a3}\Big)^2 + \frac{1}{2}\Big| D_0 \phi^i - s_2 D_3 \phi^i + s_3 i \phi^i \Big|^2 \\ & + \frac{1}{2}\Big| (D_1 + s_1 i D_2) \phi^i \Big|^2 + \frac{1}{2} (F_{03})^2 + \frac{1}{4}\Big| [\phi^i, \phi^j] \Big|^2 \\ & + s_2 \left(F_{a0} F_{a3} + \frac{1}{2}\left(D_0 \phi^i D_3 \bar{\phi}^i + D_0 \bar{\phi}^i D_3 \phi^i\right)\right) + s_3 \frac{i}{2}\left(\bar{\phi}^i D_0 \phi^i - \phi^i D_0 \bar{\phi}^i\right) \Bigg] . \end{aligned} \tag{C.9}$$

The signs (s_1, s_2, s_3) are freely chosen as either $(+,+,+)$, $(+,-,-)$, $(-,+,-)$ or $(-,-,+)$. Setting $(s_1, s_2, s_3) = (+,+,+)$ leads to equation (4.6). Other cases can be studied in a similar way.

C.2 Derivation of gauss constraint in $\mathbb{R}^4$

We start from the equation of motion

$$D^\mu F_{\mu\nu} + \frac{i}{2}\Big([\phi^i, D_\nu \bar{\phi}^i] - [D_\nu \phi^i, \bar{\phi}^i] \Big) = 0 \ . \tag{C.10}$$

Inserting $\nu = I$ and $\nu = \bar{I}$, one obtains

$$\begin{aligned} D_J F_{\bar{J}I} + \epsilon_{JI} D_{\bar{J}} F_{12} &= -\frac{i}{4}\Big([\phi^i, D_I \bar{\phi}^i] - [D_I \phi^i, \bar{\phi}^i] \Big) &\text{(C.11)}\\ D_{\bar{J}} F_{J\bar{I}} &= -\frac{i}{4}[\phi^i, D_{\bar{I}} \bar{\phi}^i] \ . &\text{(C.12)}\end{aligned}$$

Contracting $r^2 Z^I$ with the first equation, imposing a BPS equation and using the complex conjugation rule for field strengths, one obtains

$$\bar{\mathcal{D}} F_{12} - \mathcal{D}(F_{12})^\star = -\frac{i}{4}\Big(\Big[\phi^i, Z \cdot D\phi^{i\star}\Big] - \Big[Z \cdot D\phi^i, \phi^{i\star}\Big] - 2\Big[\phi^i, \phi^{i\star}\Big] \Big) \tag{C.13}$$

where the curly derivatives denote

$$\mathcal{D} \equiv \frac{1}{r^2}\left(\bar{Z}^2 D_1 - \bar{Z}^1 D_2\right) , \quad \bar{\mathcal{D}} \equiv r^2\left(Z^2 D_{\bar{1}} - Z^1 D_{\bar{2}}\right) \ . \tag{C.14}$$

This can be rewritten as

$$\left(\bar{\mathcal{D}} F_{12} + \frac{i}{4}\Big[\phi^i, Z \cdot D\phi^{i\star}\Big] - \frac{i}{4}\Big[\phi^i, \phi^{i\star}\Big]\right) - \left(\bar{\mathcal{D}} F_{12} + \frac{i}{4}\Big[\phi^i, Z \cdot D\phi^{i\star}\Big] - \frac{i}{4}\Big[\phi^i, \phi^{i\star}\Big]\right)^\star = 0 \ . \tag{C.15}$$

The second part of the Gauss constraint, (C.12), can be shown to be automatically satisfied by using the expression for $F_{I\bar{J}}$ in terms of conjugate components, (4.20), and the BPS equations $F_{I\bar{J}} \bar{Z}^J = 0$ and $D_{\bar{I}} \phi^i = 0$. Therefore, all the equations of motion are satisfied by solutions to the BPS equations except for the constraint (C.15) which comes from the Gauss law. This constraint requires that the imaginary part of $\bar{\mathcal{D}} F_{12} + \frac{i}{4}[\phi^i, Z \cdot D\phi^{i\star}] - \frac{i}{4}[\phi^i, \phi^{i\star}]$ is zero. One can actually show that this last quantity is imaginary, so that the constraint from the Gauss law can simply be written as

$$\bar{\mathcal{D}} F_{12} + \frac{i}{4}\Big[\phi^i, Z \cdot D\phi^{i\star}\Big] - \frac{i}{4}\Big[\phi^i, \phi^{i\star}\Big] = 0 \ . \tag{C.16}$$

D. Relation between spherical coordinates and complex coordinates

The spherical coordinates, (r, θ, ψ, ϕ), on $S^3 \times \mathbb{R}$ used in section 4.1 are related to the complex coordinates of sections 4.2), (4.3, 5 and 6 by:

$$\begin{aligned} Z^1 &= r\cos\zeta \ e^{i\frac{\psi+\phi}{2}} \\ Z^2 &= r\sin\zeta \ e^{i\frac{\psi-\phi}{2}} \ , \end{aligned} \tag{D.1}$$

where $\zeta = \theta/2$. This induces the following transformation of derivatives:

$$\begin{aligned}
\partial_\tau &= Z^I \partial_I + \bar{Z}^I \partial_{\bar{I}} \\
\partial_3 \equiv 2\partial_\psi &= i \left(Z^I \partial_I - \bar{Z}^I \partial_{\bar{I}} \right) \\
\partial_1 \equiv 2\partial_\theta &= -\left| \frac{Z^2}{Z^1} \right| \left(Z^1 \partial_1 + \bar{Z}^1 \partial_{\bar{1}} \right) + \left| \frac{Z^1}{Z^2} \right| \left(Z^2 \partial_2 + \bar{Z}^2 \partial_{\bar{2}} \right) \\
\partial_2 \equiv \frac{2}{\sin\theta} \left(\partial_\phi - \cos\theta \partial_\psi \right) &= -i \left| \frac{Z^2}{Z^1} \right| \left(Z^1 \partial_1 - \bar{Z}^1 \partial_{\bar{1}} \right) + i \left| \frac{Z^1}{Z^2} \right| \left(Z^2 \partial_2 - \bar{Z}^2 \partial_{\bar{2}} \right) ,
\end{aligned} \tag{D.2}$$

where $\partial_1, \partial_2, \partial_3$ are derivatives in the orthonormal frame (4.5) on S^3. We also have

$$\bar{Z} \cdot \bar{\partial} = \frac{1}{2} r \partial_r + i \partial_\psi \, , \tag{D.3}$$

which we use in section 5.1.

E. A class of exact $\mathrm{SU}(2)_L$ invariant solutions

Now we will identify a set of exact BPS solutions to the equations (4.29) and (4.30) by imposing additional symmetry. We set the scalars to zero and consider solutions which preserve an $\mathrm{SU}(2)_L \subset \mathrm{SO}(4)$ symmetry, rotating Z^1 and Z^2 as doublets. Since A_I and $A_{\bar{I}}$ should transform as fundamental and anti-fundamental representations, respectively, the only surviving terms in the series expansions are

$$A_I = iM \frac{\bar{Z}^I}{r^2} + N \epsilon_{IJ} Z^J \tag{E.1}$$

$$A_{\bar{I}} = N^* \epsilon_{\bar{I}\bar{J}} \frac{\bar{Z}^J}{r^4} \, , \tag{E.2}$$

where M, N are constant matrices with no coordinate dependence and M is required to be a Hermitian matrix.

With the scalars set to zero, the Gauss' constraint on F_{12} becomes $D_{\bar{I}} F_{12} = 0$ and substituting (E.1) and (E.2) we find

$$D_{\bar{I}} F_{12} = 0 \;\; \rightarrow \;\; \Big[N^*, 2N + [M, N] \Big] = 0 \, , \tag{E.3}$$

where we used

$$F_{12} = -2N - [M, N] \;\; . \tag{E.4}$$

Now using the expression

$$F_{I\bar{J}} = \left([M, N^*] - 2N^* \right) \frac{\bar{Z}^I (\epsilon_{\bar{J}\bar{K}}) \bar{Z}^K}{r^6} - iM \left(\frac{\delta_{I\bar{J}}}{r^2} - \frac{\bar{Z}^I Z^J}{r^4} \right) - i[N, N^*] \frac{(\epsilon_{IK} Z^K)(\epsilon_{\bar{J}\bar{L}} \bar{Z}^L)}{r^4} \tag{E.5}$$

one finds that the condition $F_{1\bar{1}} + F_{2\bar{2}} = 0$ becomes

$$0 = M + [N, N^*] \; . \tag{E.6}$$

Inserting this back into the condition (E.3) and rearranging, one finds the final algebraic conditions on the (numerical) matrices:

$$0 = M + [N, N^*] \tag{E.7}$$
$$4M = [N^*, [N, M]] + [N, [N^*, M]] , \tag{E.8}$$

where we used $[N^*, [N, M]] = [N, [N^*, M]]$ at the second equation, which can be derived from Jacobi identity and the first equation. If we identify the matrices as

$$M \sim Z^3 , \quad N \sim X^1 + iX^2 , \tag{E.9}$$

where X^a $(a = 1, 2, 3)$ are the three scalars in a BMN matrix model, these two equations are exactly the same as the ones obtained by solving the 1/16 BPS equations of BMN matrix model in [41].

We can count the degrees of freedom carried by this solution by comparing the matrix variables and the number of equations. We have $3N^2$ real numbers which should satisfy $2N^2$ equations (2 matrix equations), so we are left with N^2 real degrees of freedom. However, since we can use the $\mathrm{U}(N)$ global gauge transformation to diagonalize one of the three matrices, we have N gauge-invariant degrees left. For instance, we may take advantage of this gauge transformation to make M diagonal.

There is a curious branch of additional modes if this diagonal matrix M vanishes. Then the counting the number of equations (that is $2N^2$) in the previous paragraph does not work, since the second equation simply disappears for $M = 0$. In this case, the solution is a diagonal matrix N, which is the $\mathrm{SU}(2)_L$ invariant part of the $\mathrm{U}(1)^N$ solutions of the zero coupling limit of the gauge theory.

Collecting both branches of $\mathrm{SU}(2)_L$ invariant solutions, we have found $2N$ modes, some diagonal and others non-diagonal.

F. Appendix to section 5

F.1 Arguments establishing the obstruction at $\mathcal{O}(g_{\mathrm{YM}})$

Dividing the problem into two steps, we may rewrite equation (5.11) as:

$$h = \bar{\delta} f_1 \quad (\rightarrow \; h^\star = \delta f_1^\star) \tag{F.1}$$
$$\delta\bar{\delta} h - \bar{\delta}\delta h^\star = i \left[\bar{\delta} f_0^\star, \delta f_0\right] + 2i \left[f_0^\star, f_0\right] . \tag{F.2}$$

The equation (F.2) is a cousin of the 'Laplace' equation with source. We will state an integrability condition for this equation, which restricts the right hand side of (F.2). Since h has antiholomorphic degree zero, one finds from (5.2) that $\delta\bar{\delta}$ acts on h as a Laplacian operator $r^2 \partial_I \partial_{\bar{I}} = \frac{1}{r}\partial_r r^3 \partial_r + \nabla_{S^3}$ which is diagonalized by the functions

$$\psi^m_{s,j_1,j_2} = r^m S_s^{j_1,j_2} , \tag{F.3}$$

where $S_s^{j_1,j_2}$ are the scalar spherical harmonics of rank s in $\mathbb{R}^4$. We expand h in terms of these functions and impose the following requirements:

(A) h must be degree zero in $\bar{Z}^I$.

(B) h must belong to the image of the operator $\bar{\delta}$ since $h = \bar{\delta} f_1$.

With these conditions, we would like to see which spherical harmonics can appear on the left hand side of (F.2). The answer to this question will constrain the form of the zeroth order solution $f_0(Z)$.

We first examine the condition (A). We choose the convention in which the coordinate ψ used in section 4.1 corresponds to the angular momentum j_1 appearing in the spherical harmonics $S_s^{j_1,j_2}$ (the coordinate relations may be found in appendix D). This means that restricting to functions of antiholomorphic degree zero requires

$$\bar{Z} \cdot \bar{\partial} \psi^m_{s,j_1,j_2} = \left(\frac{1}{2} r\partial_r + i\partial_\psi\right) r^m S_s^{j_1,j_2} = (m/2 - j_1) r^m S_s^{j_1,j_2} = 0. \tag{F.4}$$

Therefore a spherical harmonic expansion of h will only contain terms of the form $r^{2j_1} S_s^{j_1,j_2}$.

Now we study the condition (B). The form $h = \bar{\delta} f_1 = r^2 \epsilon^{\bar{I}\bar{J}} Z^J \partial_{\bar{I}} f_1$ implies that all terms in h must contain at least 1 factor of either Z^1 or Z^2. This means that, in the Cartesian coordinate, terms of the form $\left(\frac{\bar{Z}^1}{r^2}\right)^c \left(\frac{\bar{Z}^2}{r^2}\right)^d$ do not appear in h. All other terms can appear in h.

With h satisfying the conditions (A) and (B), we want to know what kind terms in the spherical harmonic expansion appear on the left hand side of (F.2). Since the eigenvalue of the Laplacian operating on $\psi^m_{s,j_1,j_2} = r^m S_s^{j_1,j_2}$ is $m(m+2) - s(s+2)$, the modes in h which have $m = s$ or $m = -(s+2)$ are annihilated by $\delta\bar{\delta}$ and do not appear in $\delta\bar{\delta}h$. Now we see the following:

1. From condition (A), $m = 2j_1$, so harmonics with $m = -(s+2)$ cannot appear in h at all since $|2j_1| \le s$. We therefore conclude that it is the harmonics with $m = s$ that are constrained not to appear in $\delta\bar{\delta}h$. In Cartesian coordinates, these modes $\psi^s_{s,j_1,j_2} = r^s S_s^{s/2,j_2}$ in h correspond to purely holomorphic terms, $(Z^1)^a (Z^2)^b$.

2. From condition (B), the terms $\left(\frac{\bar{Z}^1}{r^2}\right)^c \left(\frac{\bar{Z}^2}{r^2}\right)^d$ do not appear in h and therefore they cannot appear in $\delta\bar{\delta}h$.

All other ψ^m_{s,j_1,j_2} which appear in a spherical harmonic expansion of h are not annihilated by the Laplacian $r^2 \partial_I \partial_{\bar{I}}$ and may appear in $\delta\bar{\delta}h$. In summary, all terms $(Z^1)^a (Z^2)^b \left(\frac{\bar{Z}^1}{r^2}\right)^c \left(\frac{\bar{Z}^2}{r^2}\right)^d$ may appear in $\delta\bar{\delta}h$ except purely holomorphic terms ($c = d = 0$), and their $\star$ conjugates ($a = b = 0$). Since $\bar{\delta}\delta h^\star = (\delta\bar{\delta}h)^\star$, the same constraints apply to the term $\bar{\delta}\delta h$ on the left hand side of equation (F.2).

F.2 Integrability constraint without scalars

In this appendix, we explicitly extract the holomorphic portions of the right hand side of equation (F.2). We use the notation of section 5.1 and first note a useful integral:

$$\int_{r=1} d^3\Omega Y_{k_1+n_1,k_2+n_2}(Y_{k_1,k_2})^\star (Y_{n_1,n_2})^\star \tag{F.5}$$
$$= \sqrt{\frac{(n_1+n_2+1)!(k_1+k_2+1)!(n_1+n_2+k_1+k_2+1)!}{(2\pi^2)^3 n_1!n_2!k_1!k_2!(n_1+k_1)!(n_2+k_2)!}} \int_{r=1} d^3\Omega |Z^1|^{2(n_1+k_1)}|Z^2|^{2(n_2+k_2)}$$
$$= \sqrt{\frac{1}{2\pi^2}\cdot\frac{(n_1+n_2+1)!(k_1+k_2+1)!}{n_1!n_2!k_1!k_2!}\cdot\frac{(n_1+k_1)!(n_2+k_2)!}{(n_1+n_2+k_1+k_2+1)!}} = \frac{C_{n_1n_2}C_{k_1k_2}}{C_{n_1+k_1,n_2+k_2}} \equiv c^{n_1n_2}_{k_1k_2} \ .$$

This integral allows us to extract the holomorphic component of $2[f_0^\star, f_0]$, which is

$$2[f_0^\star, f_0] \to \sum_{n_1,n_2=0}^{\infty} Y_{n_1n_2}(Z^1,Z^2) \sum_{k_2,k_2=0}^{\infty} 2c^{n_1n_2}_{k_1k_2}\left[a^*_{k_1k_2},\, a_{n_1+k_1,n_2+k_2}\right] \ . \tag{F.6}$$

We can extract the holomorphic terms of $\left[\,\bar\delta f_0^\star, \delta f_0\right]$ in a similar manner. Acting the operator δ on $Y_{n_1n_2}$, one obtains

$$\delta Y_{k_1k_2} = \sqrt{k_1(k_1+k_2+1)}\,\frac{\bar Z^2}{r^2}\,Y_{k_1-1,k_2} - \sqrt{k_2(k_1+k_2+1)}\,\frac{\bar Z^1}{r^2}\,Y_{k_1,k_2-1} \ . \tag{F.7}$$

The coefficients of the holomorphic term proportional to $Y_{n_1n_2}$ can be obtained by observing that the following integral gives the only nonzero contribution:

$$\begin{aligned}
&\int_{r=1} d^3\Omega\,(\delta Y_{k_1+n_1,k_2+n_2})\,(\delta Y_{k_1,k_2})^\star(Y_{n_1,n_2})^\star \\
&= C_{n_1n_2}C_{k_1k_2}C_{n_1+k_1,n_2+k_2}\int_{r=1} d^3\Omega\,\Big(\,k_1(k_1+n_1)|Z^1|^{2(n_1+k_1-1)}|Z^2|^{2(n_2+k_2+1)} \\
&\qquad\qquad +k_2(k_2+n_2)|Z^1|^{2(n_1+k_1+1)}|Z^2|^{2(n_2+k_2-1)} \\
&\qquad\qquad -\Big(k_1(k_2+n_2)+k_2(k_1+n_1)\Big)\,|Z^1|^{2(n_1+k_1)}|Z^2|^{2(n_2+k_2)}\Big) \\
&= C_{n_1n_2}C_{k_1k_2}C_{n_1+k_1,n_2+k_2}\Bigg(\frac{k_1(k_1+n_1)}{(C_{n_1+k_1-1,n_2+k_2+1})^2} \\
&\qquad\qquad +\frac{k_2(k_2+n_2)}{(C_{n_1+k_1+1,n_2+k_2-1})^2} - \frac{k_1(k_2+n_2)+k_2(k_1+n_1)}{(C_{n_1+k_1,n_2+k_2})^2}\Bigg) \\
&= c^{n_1n_2}_{k_1k_2}\Big(\,k_1(n_2+k_2+1)+k_2(n_1+k_1+1) - \Big(k_1(k_2+n_2)+k_2(k_1+n_1)\,\Big)\Big) \\
&= (k_1+k_2)c^{n_1n_2}_{k_1k_2} \ .
\end{aligned} \tag{F.8}$$

Note that the above coefficient is just zero for the case $(k_1,k_2)=(0,0)$. With this integral in hand, we collect all the holomorphic parts of $\left[\,\bar\delta f_0^\star, \delta f_0\right] + 2[f_0^\star, f_0]$,

$$\begin{aligned}
&\left[\bar\delta f_0^\star, \delta f_0\right] + 2[f_0^\star, f_0] \\
&\quad \to \sum_{n_1,n_2=0}^{\infty} Y_{n_1n_2}(Z^1,Z^2)\sum_{k_1,k_2=0}^{\infty}(k_1+k_2+2)\,c^{n_1n_2}_{k_1k_2}\left[a^*_{k_1k_2},\, a_{n_1+k_1,n_2+k_2}\right] \\
&\quad = \sum_{n_1,n_2=0}^{\infty} Y_{n_1n_2}(Z^1,Z^2)\sum_{k_1,k_2=0}^{\infty}(k_1+k_2+2)\,\frac{C_{n_1n_2}C_{k_1k_2}}{C_{n_1+k_1,n_2+k_2}}\left[a^*_{k_1k_2},\, a_{n_1+k_1,n_2+k_2}\right] \ ,
\end{aligned} \tag{F.9}$$

which is the result that appears in equation (5.16).

F.3 Integrability constraint with scalars

In this appendix, we extract the purely holomorphic terms from the source appearing on the right hand side of equation (5.26) in the most general bosonic case, where all scalars and the gauge field may be nonzero. The treatment of the $[\phi_0^{i\star}, \phi_0^i] + [\bar{\delta}\phi_0^{i\star}, \delta\phi_0^i]$ part of the source is very similar to that of appendix F.2. It gives the following holomorphic terms

$$\begin{aligned}
&\left[\bar{\delta} f_0^\star, \delta f_0\right] + 2[f_0^\star, f_0] + \frac{1}{4}\left([\phi_0^{i\star}, \phi_0^i] + [\bar{\delta}\phi_0^{i\star}, \delta\phi_0^i]\right) \\
&\quad \to \sum_{n_1,n_2=0}^{\infty} Y_{n_1 n_2} \sum_{k_1,k_2=0}^{\infty} c_{k_1 k_2}^{n_1 n_2} \Bigg((k_1+k_2+2)\left[a_{k_1 k_2}^*,\ a_{n_1+k_1, n_2+k_2}\right] \\
&\qquad\qquad + \frac{1}{4}(k_1+k_2+1)\left[b_{k_1 k_2}^{i*},\ b_{n_1+k_1, n_2+k_2}^i\right]\Bigg)
\end{aligned} \tag{F.10}$$

where we are using the same notation as in section 5.1. The remaining term in the source takes the form $(\delta\bar{\delta} + \bar{\delta}\delta)(\cdots)$, where

$$\delta\bar{\delta} + \bar{\delta}\delta \stackrel{eff}{=} 2r^2 \partial_I \partial_{\bar{I}} - (Z \cdot \partial) \ . \tag{F.11}$$

We now collect the purely holomorphic portions of this term.

$$\frac{i}{4}\left(\delta\bar{\delta} + \bar{\delta}\delta\right)\left[\phi_0^i, \phi_0^{i\star}\right]' = \frac{i}{4}\left(2r^2 \partial_I \partial_{\bar{I}} - (Z \cdot \partial)\right)\left[\phi_0^i, \phi_0^{i\star}\right]' = F(Z) + F^\star\left(\frac{\bar{Z}}{r^2}\right) , \tag{F.12}$$

where the prime denotes only keeping the terms in $[\phi, \phi^\star]$ which would give holomorphic plus conjugate terms in the right hand side. We have seen in section 5.1 that $\partial_I \partial_{\bar{I}}$ cannot produce any such terms and therefore the holomorphic part of the $\delta\bar{\delta} + \bar{\delta}\delta$ contribution to the source is simply the holomorphic part of

$$-\frac{i}{4}(Z \cdot \partial)\left[\phi_0^i, \phi_0^{i\star}\right] \ . \tag{F.13}$$

Collecting all contributions, we find that the holomorphic part of the source is

$$\begin{aligned}
&\left[\bar{\delta} f_0^\star, \delta f_0\right] + 2[f_0^\star, f_0] + \frac{1}{4}\left(\left[\phi_0^{i\star}, \phi_0^i\right] + \left[\bar{\delta}\phi_0^{i\star}, \delta\phi_0^i\right]\right) + \frac{1}{4}(Z \cdot \partial)\left[\phi_0^{i\star}, \phi_0^i\right] \ . \\
&\quad \to \sum_{n_1,n_2=0}^{\infty} Y_{n_1 n_2} \sum_{k_1,k_2=0}^{\infty} \times\, c_{k_1 k_2}^{n_1 n_2} \Bigg((k_1+k_2+2)\left[a_{k_1 k_2}^*,\ a_{n_1+k_1, n_2+k_2}\right] \\
&\qquad\qquad + \frac{1}{4}(n_1+n_2+k_1+k_2+1)\left[b_{k_1 k_2}^{i*},\ b_{n_1+k_1, n_2+k_2}^i\right]\Bigg)
\end{aligned} \tag{F.14}$$

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